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THE EXCITATION OF THE ENHANCED SPECTRA OF SODIUM AND POTASSIUM IN A LOW VOLTAGE ARC[1]

By PAUL D. FOOTE, W. F. MEGGERS, and F. L. MOHLER

ABSTRACT

Variation of the spectra of Na and K with energy of electronic impact.—The quantum theory, as developed by Bohr and Sommerfeld, leads to the conclusions: that all the arc lines are emitted by neutral atoms and should be excited at voltages above the ionization voltages, 5.1 for Na and 4.3 for K, and that the enhanced lines are emitted by singly ionized atoms and, in the case of the alkalies, should be excited when (*b*) the valence electron is knocked out of its orbit, after (*a*) an electron from the outer ring (*L*-ring for Na, *M*-ring for K) has been removed. In the case of Na, the ionization (*a*) requires 35 volts if it is complete, giving *La*-radiation, or 30 volts if the electron goes to the $L\alpha$ orbit; and since the displacement (*b*) requires a maximum of 14 volts, the enhanced spectrum for Na should be excited by two successive impacts at 35 volts and possibly at 30 volts, and the highest frequency should correspond to about 14 volts. The corresponding figures for K are: 23 volts for the excitation of the *Ma*-radiation, possibly 20 volts for $M\alpha$ and the enhanced spectrum, and 11 volts for the energy corresponding to the highest frequency. Since these conclusions contradict the general belief that high potentials are necessary to excite these enhanced spectra, a series of *spectrograms* was made, with a large quartz spectrograph, of the light excited by low voltage electrons in a special cylindrical tube in which the distance between the central tungsten filament, which supplied the electrons, and the accelerating grid was so short and the vapor pressure used was so low (0.1 to 0.2 mm of Hg) that practically all the electrons entered the force-free space surrounding the grid, with the same energy. The voltages used varied from 3.5 to 5000 for Na, and from 3.5 to 40 for K. The results completely verify the theoretical conclusions. Tables and photographs are given which clearly show the *three-stage development* in the spectra of these elements: (1) Single pair stage: 2.1 to 5.1 volts (Na); 1.6 to 4.3 volts (K). (2) Arc spectrum stage: 5.1 to 30 volts (Na); 4.3 to 20 volts (K). (3) Enhanced spectrum stage: above 30 volts (Na); above 20 volts (K). With a heavy current (1 ampere) at 7 volts in K, the pair λ 4642 becomes very prominent, though it has the

[1] Published by permission of S. W. Stratton, Director, Bureau of Standards.

series notation ($1s-3d$). Its presence cannot be due to incipient Stark effect, and accordingly furnishes an exception to the principle of selection as applied by Sommerfeld. The *wave-lengths in I.A. and the series notations,* when known, are given for 250 lines.

The quantum theory of spectra attributes enhanced lines to ionized atoms. In the case of simply ionized helium the Bohr theory, as generalized by Sommerfeld, not only gives the correct wave-lengths of the observed enhanced lines, but their fine structure as well.

In applying the quantum theory to heavier atoms certain simplifying assumptions must be made in order to carry out the mathematical analysis. Thus Sommerfeld[1] considers the case where the atom has one electron in an outer orbit and where as an approximation the other electrons may be treated as though lying in a circular coplanar orbit. These electrons, as far as their action on the outer orbit is concerned, may be assumed equivalent to a ring of electricity of equal total charge.

Evaluating the kinetic energy of the outer electron, and applying the integral $\int pdq = nh$ both in respect to azimuth and radius as independent variables, in accordance with the quantum theory, Sommerfeld derives the following expression for the variable term of a spectral series, identical in form with the familiar Ritz equation:

$$(m,a) = \frac{N(E/e)^2}{[n_a + n_r + a - \alpha(m,a)]^2} \qquad (1)$$

where E is the excess in charge of the core over that of the ring, n_a the azimuthal and n_r the radial quantum numbers characterizing the outer orbits, and a and α constants which are approximately proportional to E and are functions of n_a but not of n_r.

An atom of an alkali metal has one electron in its outer orbit while that of an alkali earth has two, accounting for the monovalent and bivalent properties respectively For the neutral atom of an alkali metal $(E/e)^2 = 1$, and equation (1) represents the variable terms of the arc spectra. If one of the outer electrons of an atom of an alkali earth is permanently removed, that is, if the atom is simply ionized, equation (1) likewise applies except that $(E/e)^2 = 4$ and the values of a and α are altered in a definite manner.

[1] *Atombau* (2d ed.), p. 506.

Hence if the enhanced spectrum of magnesium is due to the simply ionized atom, it should resemble the arc spectrum of sodium. Such is the case, even to the appearance approximately of $4N$ and $a^* = 1.5\, a$ instead of N and a in the spectral formulae representing the enhanced lines.[1] A similar relation exists between the spectra of the other elements of these two groups. From the above and further considerations Kossel and Sommerfeld[2] concluded that in so far as enhanced spectra arise in *simply* ionized atoms the enhanced spectrum of any element should resemble the arc spectrum of the element of next lower atomic number. Very likely higher types of enhanced spectra may be produced by atoms which have suffered multiple ionization.

Now, as is well recognized, the complete arc spectrum of any monatomic vapor may be excited by electronic collision with a normal atom when the energy of the impacting electron is eV, where V is the ionization potential. Furthermore for all monatomic vapors not exhibiting metastable modifications[3] the voltage V is related to the highest convergence frequency ν of the arc spectrum by the quantum condition $ch\nu = eV$. The highest frequency in the arc spectra of the alkali earths is $1S$ and of the alkali metals $1s$, corresponding to ionization potentials from 4 to 10 volts. Similarly in the enhanced spectrum of an alkali earth, the highest frequency is $1\mathfrak{S}$, and accordingly at the corresponding voltage V^* the complete simple enhanced spectrum should be excited. We have verified this experimentally for magnesium.[4]

[1] Sommerfeld neglected to consider the shrinkage of the ring when the number of electrons in the ring is maintained constant while the charge on the core is increased by one unit. If this is taken into account we find for sodium and potassium $a^* = 1.5\, a$ instead of $a^* = 2\, a$ given by Sommerfeld. The former relation is well satisfied if we assign values of 1, 2, 3, etc., instead of 1.5, 2.5, etc., to m in the (ms) terms. Cf. *Journal of the Optical Society of America and Review of Scientific Instruments*, January, 1922.

[2] *Verh. d. Phys. Ges.*, **21**, 240, 1919.

[3] From spectroscopic and other considerations there appear to be two modifications of helium, possibly one in which the two electrons are in crossed orbits and the other in which the orbits are coplanar. In polyatomic vapors the ionization potential may involve the work of dissociation in addition to the ionization of one or more atoms.

[4] The enhanced spectrum of magnesium is produced by electronic impact of 14.97 volts corresponding to the wave-number $1\mathfrak{S} = 121{,}270$, in which case the impact occurs with a simply ionized atom, with the result that the second valence electron

These voltages again are all small, of the order of 10 to 20 volts for the elements in this group of the periodic table.

The simple enhanced spectra of the alkali metals should resemble the complicated arc spectra of the rare gases. Since only recently attempts to correlate the latter in series[1] have been successful it is not surprising that nothing is known of series relations in the enhanced spectra of the alkali metals.

It is possible, however, to estimate roughly the voltages at which the enhanced spectra of sodium and potassium should be excited. In the case of sodium we have the K-ring with 2 electrons, the L-ring with 8 electrons, and the outer ring with the valence electron. There are two direct ways in which the atom may be ionized. First, the valence electron may be removed requiring an amount of work equivalent to 5.1 volts;[2] then a second collision of X volts may expel an electron from the L-ring. Secondly, an electron may be ejected from the L-ring requiring 35 volts,[3] followed by a collision of Y volts expelling the valence electron. X must be somewhat greater than 35, but, since the orbit of the valence electron is much larger than that of the L-ring, it is not evident that the repulsive force of the outer electron can materially assist in the expulsion of an electron from the L-ring. Hence to doubly ionize sodium by the first method should require electronic impacts of maximum value $La+\delta La$ volts where δLa is a small quantity and δ is a factor. The total work of double ionization is then $(La+\delta La+5)$ volts. In the second method of ionization accordingly $Y=\delta La+5$ which must be less than 35 volts. Hence by the first method of excitation, the enhanced spectrum of sodium should appear at $35(1+\delta)$ volts and by the second method at 35 volts, and in both cases should accompany its L-radiation.

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is removed. This paper, *Phil. Mag.*, **42,** 1002, December, 1921, is of especial interest since De Gramont and Hemsalech, *C.R.*, **173,** 505, 1921, have just published a report of experiments in which they found that a potential gradient of 500 volts/cm was required to excite the enhanced spectrum of magnesium.

[1] For example, neon: Paschen, *Annalen der Physik*, **60,** 405, 1919.

[2] Tate and Foote, *Phil. Mag.*, **36,** 64, 1918.

[3] Mohler and Foote, "Soft X-Rays from Arcs in Vapors," *Journal of the Optical Society of America*, **5,** 328, 1921. For sodium $La=35$ volts.

Similar considerations for potassium show that its enhanced spectrum should be excited by direct impact at about 20 volts and should accompany its M-radiation.[1]

It is possible to arrive at an approximate estimate of the value of δ in the foregoing formulae. After expulsion of an electron from the L-ring, the ring shrinks, and provided the valence electron does not fall into the L-ring—a condition which would curtail the emission of the enhanced lines—it assumes a new orbit where the angular momentum is unity. This orbit represents a higher energy level. Following Sommerfeld's method of derivation except that two coplanar rings of electrons are assumed and their shrinkage is given consideration, we find for sodium and potassium:

$$\left.\begin{array}{ll} a^{*}=\dfrac{2(r+x)}{r+y}\,a & x=\dfrac{112}{(9-s_7)^2} \\ r=\dfrac{Z-9}{(Z-s_{Z-9})^2} & y=\dfrac{128}{(9-s_8)^2} \end{array}\right\} \tag{2}$$

where a^* and a are the values of a in the Ritz formula, equation (1), for the enhanced and arc lines respectively, Z the atomic number and s_n the nuclear defect of the ring, i.e., effect of the interaction of the electrons. To an approximation, omitting terms beyond a in the Ritz formula:

$$V=\frac{13.55}{(1+a)^2} \tag{3}$$

$$V^*=\frac{54}{(1+a^*)^2} \tag{4}$$

where V is the simple ionization potential and V^* the voltage corresponding to the work required to remove the valence electron from its new quantized orbit, after the removal of an electron from the L-ring. V^* accordingly corresponds to the highest convergence frequency in the spark spectrum.

Equation (2) gives $a^*=1.5\,a$ for sodium and potassium. Computing a from equation (3) and the known ionization potentials, we obtain from (4) the following:

$$V^*=14 \text{ for sodium}$$

$$V^*=11.5 \text{ for potassium.}$$

[1] Mohler and Foote, *loc. cit.*, observed two M-limits for potassium one at 20 volts and one at 23 volts.

The total work of double ionization is accordingly:

$$\text{Total work} = La + V^* = 35 + 14 = 49 \text{ volts for sodium}$$
$$= 23 + 11 = 34 \text{ volts for potassium.}$$

Hence for sodium

$$La + \delta La + V = 49$$
$$\delta La = 9$$
$$\delta = 0.3.$$

Similarly for potassium

$$\delta = 0.3.$$

These values of δ confirm the estimates made above from more qualitative considerations. It is of interest to note that in sodium, for example, although it requires but 14 volts to excite the enhanced spectrum, this spectrum cannot be produced at this voltage since as a prerequisite the displacement of an electron from the L-ring is essential, which necessitates electronic impact of higher energy value. This is in contradistinction to the phenomena observed in the case of the alkali earths. Here the removal of the valence electron at the simple ionization potential leaves the atom in the fundamental state for the emission of enhanced lines (just as the normal, unexcited atom is in the fundamental state for the emission of arc lines) and further impacts of even less energy value are sufficient to excite the "single-line" *enhanced* spectrum $1\mathfrak{S} - 2\mathfrak{P}$. However, with the alkalis, the simple ionization does not leave the atom in the fundamental state for the emission of enhanced lines, but rather in the state for the emission and even absorption of $L\alpha$ and other X-ray lines.

In the X-ray spectra of elements of higher atomic number it is known that $K\alpha$ cannot be produced except when the limit Ka is reached, under which condition all K-lines appear simultaneously. If a similar condition held for the L-radiation of sodium, it would be impossible to excite the enhanced spectrum below the voltage corresponding to the limit $La = 35$ volts. However, although a single-line X-ray spectrum has never been observed, it must be a possible phenomenon for elements of low atomic number where the ring concerned is the outermost "saturated" orbit. Hence $L\alpha = 35 - V = 30$ volts might be produced at 30 volts instead of at

$La = 35$. Such being the case the enhanced spectrum could be excited by successive impact at the voltage corresponding to *La*, the first impact (or equivalent absorption of radiation) displacing the electron to the *La* orbit, where a second impact is more than sufficient to doubly ionize. Still further, Mohler and Foote observed a new series of critical potentials of lower value than *La*, for several of the elements of low atomic number, which may be related to the *L*-series. If so, the enhanced spectra of sodium might be excited by successive impact in an arc at the voltage corresponding to this new limit.

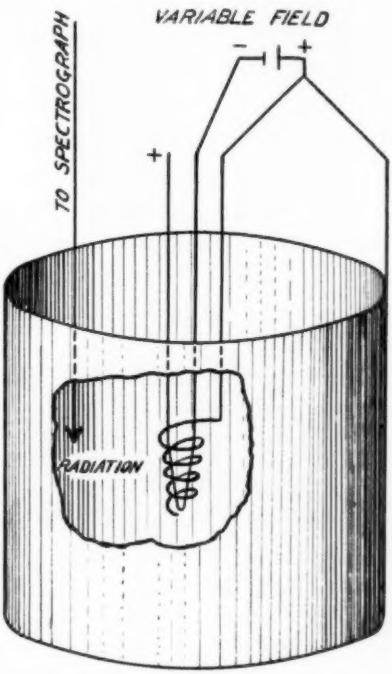


FIG. 1.—Discharge tube. The electrons attain the full velocity of the impressed field.

In view of the fact that heretofore all investigators have employed and believed requisite an exciting field of many thousand volts for the production of the enhanced spectra of sodium and potassium, the low values above estimated necessitated experimental verification.

The form of the discharge tube employed is illustrated by Figure 1. Its novelty consists in the use of a grid (helical coil) mounted extremely close to a tungsten hot wire cathode (or in some cases an equipotential surface heated inside) and in metallic contact with a concentric hollow cyclinder of relatively large diameter. By properly regulating the temperature of the sodium or potassium vapor, a vapor pressure may be maintained for which relatively few electronic-atomic collisions occur over the short accelerating field between the cathode and grid, and the majority of electrons falling into the large force-free space between the grid

and plate have, at the instant of collision, a velocity corresponding to the impressed field. The gas pressure registered by the gage was less than 0.001 mm of mercury and the vapor pressure was of the order 0.1 to 0.2 mm of mercury.

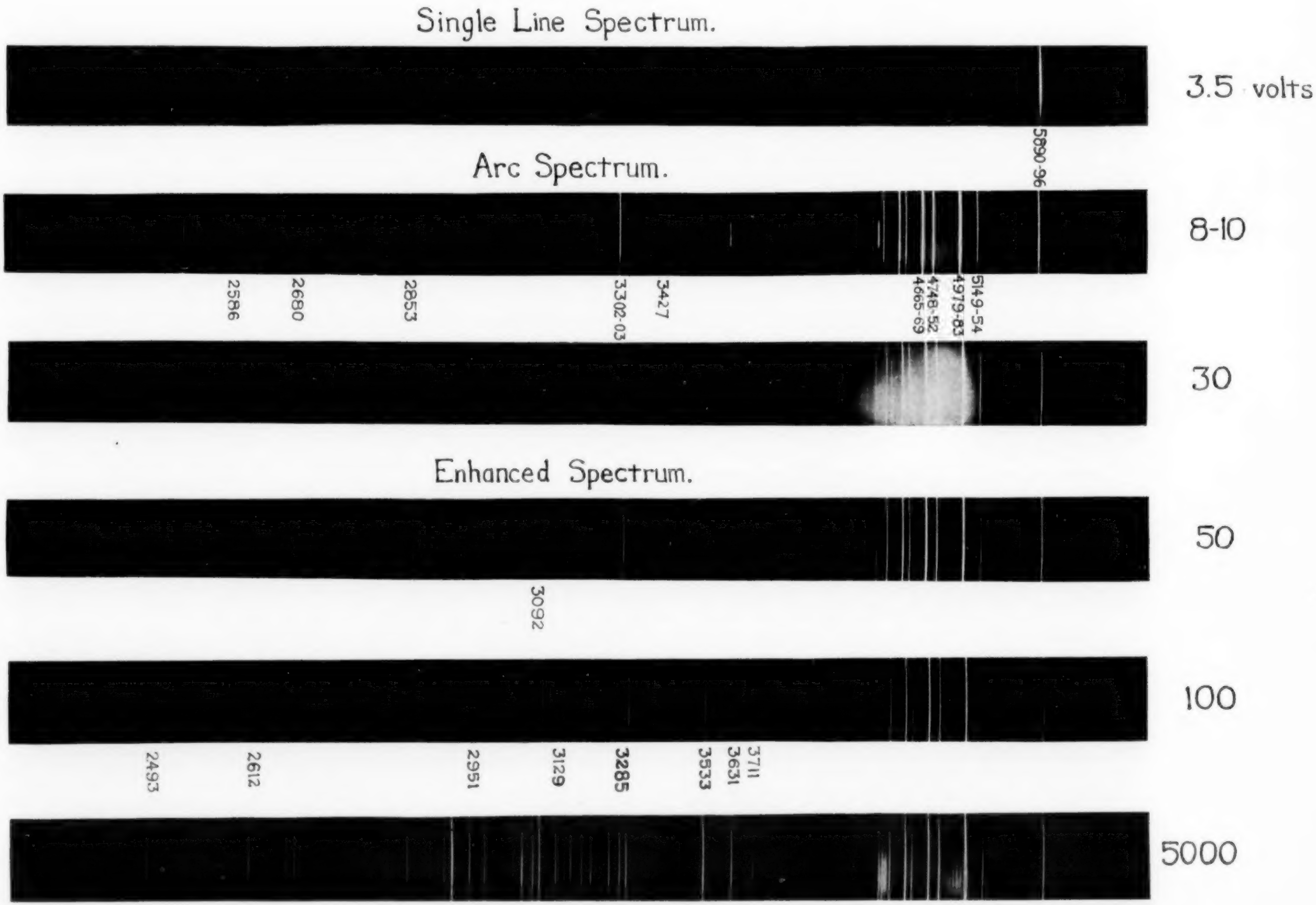
The arcs at various exciting voltages were photographed with a large Hilger Type C quartz-spectrograph. Plate II shows the results for sodium and Plate III for potassium. Ordinary Seed 30 plates were used for all exposures except the 3.5 volt potassium arc in which case the plate was made red sensitive by bathing in dicyanin. The pair λ 7699–7664, although present in the spectra at the higher voltages, was not recorded by the "blue-sensitive" plates used for the other spectrograms. The lines on all the plates were carefully measured and reduced to wave-lengths by the method of the Hartmann-Cornu interpolation formula, using lines of a mercury comparison spectrum and known arc and spark lines as normals.

SODIUM

Plate II shows the three successive stages in the sodium spectrum. The single-line spectrum consisting of the pair D_1 and D_2 is excited above the resonance potential, 2.1 volts, and below the ionization potential 5.1 volts. From 5.1 volts to about 30 volts the arc lines appear alone. At 30 volts numerous spark lines begin to appear while above 40 volts the complete enhanced spectrum is excited.

The lower spectrogram was obtained by connecting a small 5000-volt transformer between the hot wire and grid. The vapor was too rare for a discharge to pass except when the cathode was heated to give a copious emission of electrons. Since the effective resistance of the discharge gap must have been small compared to the impedance of the transformer, the actual applied voltage was undoubtedly much less than 5000 volts. Very few enhanced lines not excited at 50 volts were observed in this source but their intensity was considerably greater. All of the spectrograms of sodium were made with a very narrow slit and much of the detail readily apparent on the negatives is obscured in the reproduction. On this account reference must be made to Table I for a more careful consideration of the excitation of the enhanced lines, and verification of the foregoing statements.

PLATE II



Spectra of Sodium

TABLE I

SPECTRA OF SODIUM

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
2318.0	1−	1−	...	...	...	...
2475.51	...	...	...	...	1−	$1s-10p_{1,2}$
90.61	...	...	...	1−	1−	$1s-9p_{1,2}$
2493.36	4	4	2	1	...	...
2497.3	2	...	...	1	...	...
2506.40	1	1	...	...	...	...
12.14	1−	1−	1−	1	1	$1s-8p_{1,2}$
15.50	1	1	...	...	...	...
31.54	3	2	1	...	...	...
43.80	1−	1	1	2	2	$1s-7p_{1,2}$
86.27	1	1	...	...	...	...
93.89	1	2	2	3	3	$1s-6p_{1,2}$
2595.0	1−	1−	...	...	...	...
2612.18	5	4	1	1	...	...
61.72	4	3	1	1	...	...
71.94	5	3	1	1	...	...
78.2	3	2	1	...	...	...
2680.37	1	3	3	4	3	$1s-5p_{1,2}$
2809.76	4	3	1	1	...	...
18.5	1	1	...	...	...	...
30.0	1	1	...	...	...	...
39.36	1	1	...	...	...	...
41.99	6	4	2	...	...	...
52.80	3	4	4	5	5	$1s-4p_{1,2}$
59.56	3	2	1	...	...	...
61.2	1−	1−	...	...	...	...
70.89 } 71.39 }	3	2	1	...	...	...
73.0	1−	1−	...	...	...	...
81.33	4	3	2	1	...	...
86.46	3	2	1	...	...	...
2894.15	4	3	1	1	...	...
2901.39	3	2	1	...	...	...
05.16	6	4	3	2	...	...
17.76	4	3	3	2	...	...
19.46	3	3	2	1	...	...
21.14	3	3	2	1	...	...
23.65	2	2	1	...	...	...
31.44	1−	1−	...	...	...	...
34.42	1	1	...	...	...	...
2938.11	3	2	1	...	...	...
45.83	2	1	...	...	...	...
47.51	3	2	1	...	...	...
51.53	8	5	2	2	...	...
60.10	1	1	1−	...	...	...

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
71.1	1−	1−	....	....	....	....
75.20	5	3	2	1	....	....
77.52	1	1	....	....	....	....
79.92	7	4	2	1	....	....
2984.44	6	4	2	1	....	....
3007.71	2	2	1	....	....	....
15.81	3	2	1	1	....	....
29.70	3	3	1	1	....	....
37.29	4	3	1	1	....	....
50.48	3	1	1−	....	....	
53.97	4	3	1	....	....	....
56.34	7	5	3	2	....	....
61.75	1−	....	....	....	....	....
64.8	2	1	1−	....	....	....
66.74	2	2	1−	....	....	....
74.63	4	3	1	1	....	....
78.51	5	4	3	....	....	....
80.58	2	2	....	....	....	....
92.91	12	8	6	3	....	....
3104.6	2	2	1	....	....	....
07.6	1−	1−	....	....	....	....
24.63	2	2	1−	....	....	....
29.57	7	5	4	2	....	....
35.65	5	4	3	1	....	....
38.11	1	1	....	....	....	....
46.07	1	2	....	....	....	....
49.48	5	4	3	1	....	....
64.11	7	5	2	2	....	....
75.38	1	1	....	....	....	....
79.19	3	2	....	....	....	....
3189.94	7	5	3	2	....	....
3212.42	7	5	3	2	....	....
16.5	1	1	....	....	....	....
26.1	2	2	....	....	....	....
35.1	3	2	1−	....	....	....
51.3	1	1	....	....	....	....
58.36	7	4	2	1	....	....
60.4	1	1	....	....	....	....
74.27	3	2	....	....	....	....
85.76	8	7	4	2	....	....
3302.33	3	4	3	4	10	$1s-3p_1$
02.93	2	3	2	3	8	$1s-3p_2$
05.2	2	2	....	....	....	....
18.2	3	2	....	....	....	....
20.7	1−	1	....	....	....	....

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
3327.9	3	2	...	...	...	...
3400.2	1−	1−	...	...	...	...
27.1	1	1	1	4	7	$1s-3d$
3462.58	2	2	1	...	...	...
3533.08	12	8	5	3	...	...
3631.31	5	4	2	1	...	...
34.3	1	1	...	...	...	...
3711.15	4	3	1	...	...	...
3745.6	1−	1−	...	...	...	...
3881.0	1	1−	...	...	...	...
4124.0	2	2	...	...	...	...
4141.0	...	...	...	...	1−, *d*	$2p-18d$
48.0	...	...	...	...	1−, *d*	$2p-17d$
56.0	...	...	...	...	1−, *d*	$2p-16d$
65.0	...	...	...	...	1−	$2p_2-15d$
67.8	...	...	...	...	1−	$2p_1-15d$
77.4	...	...	...	...	1−	$2p_2-14d$
80.2	...	1−	...	1−	1	$2p_1-14d$
92.6	...	...	...	...	1−	$2p_2-13d$
4195.5	...	1	...	1	1	$2p_1-13d$
4212.4	...	...	...	...	1	$2p_2-12d$
15.4	...	1−	1−	1	2−	$2p_1-12d$
22.8	...	...	...	...	1, *d*	$2p-11s$
38.6	1−	...	...	...	1	$2p_2-11d$
41.6	1−	1	1	1	2	$2p_1-11d$
49.4	...	...	...	...	1−	$2p_2-10s$
52.4	...	1−	1−	1−	1−	$2p_1-10s$
73.2	1−	...	...	...	2	$2p_2-10d$
4276.3	1	1	1	1	3	$2p_1-10d$
87.5	...	1−	1−	1−	1−	$2p_2-9s$
4290.6	1−	1	1	1	1	$2p_1-9s$
4321.1	1	1	1	1	3	$2p_2-9d$
24.3	2	2	2	2	4	$2p_1-9d$
41.3	...	1	1	1	1	$2p_2-8s$
44.5	1	1	1	1	2	$2p_1-8s$
90.14	3	2	2	2	4	$2p_2-8d$
4393.45	4	3	3	3	6	$2p_1-8d$
4419.94	1−	1	1	1	2−	$2p_2-7s$
23.31	1	2−	2−	2−	3	$2p_1-7s$
94.266	5	4	4	4	5	$2p_2-7d$
97.724	6	5	5	5	7	$2p_1-7d$
4541.671	2	1	1	1	2	$2p_2-6s$
4545.218	3	2	2	2	4	$2p_1-6s$
4664.858	7	4	4	4	7	$2p_2-6d$
4668.597	9	6	6	6	9	$2p_1-6d$

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
4748.016	4	2	2	2	3	$2p_2-5s$
4751.891	5	2	2	2	4	$2p_1-5s$
4978.608	9	6	6	7	9	$2p_2-5d$
4982.864	10	8	8	9	10	$2p_1-5d$
5149.090	2	2	2	2	5	$2p_2-4s$
5153.645	3	3	3	3	6	$2p_1-4s$
5682.675	1	1	1	1	2	$2p_2-4d$
5688.222	1	1	1	1	2	$2p_1-4d$
5889.965	15	10	10	8	15	$1s-2p_1$
5895.932	10	8	8	6	10	$1s-2p_2$
6154.214	1	....	....	....	1	$2p_2-3s$
6160.725	1	....	....	....	1	$2p_1-3s$

DATA FOR TABLE I

Exposure Number	Volts	Milliamperes	Time of Exposure
23	5000	....	4 hours
16	100	150	30 min.
17	50	40–140	50 min.
21	30	65	65 min.
20	8–10	2000	125 min.

The wave-lengths in Table I are in international angstrom units. The spark lines given by Schillinger,[1] having been corrected from the Rowland to the international scale, are represented by six significant figures while the values for the arc lines from λ 4390.14 to λ 6160.725 are those recently measured by Datta.[2] The remaining values are given to four or five figures and are the means of several determinations from the quartz prism spectrograms; they are probably not in error by more than one unit in the last place. Approximate values for the wave-lengths of many subordinate series lines were given by Zickendraht[3] and our numbers are in fair agreement with his for the higher terms.

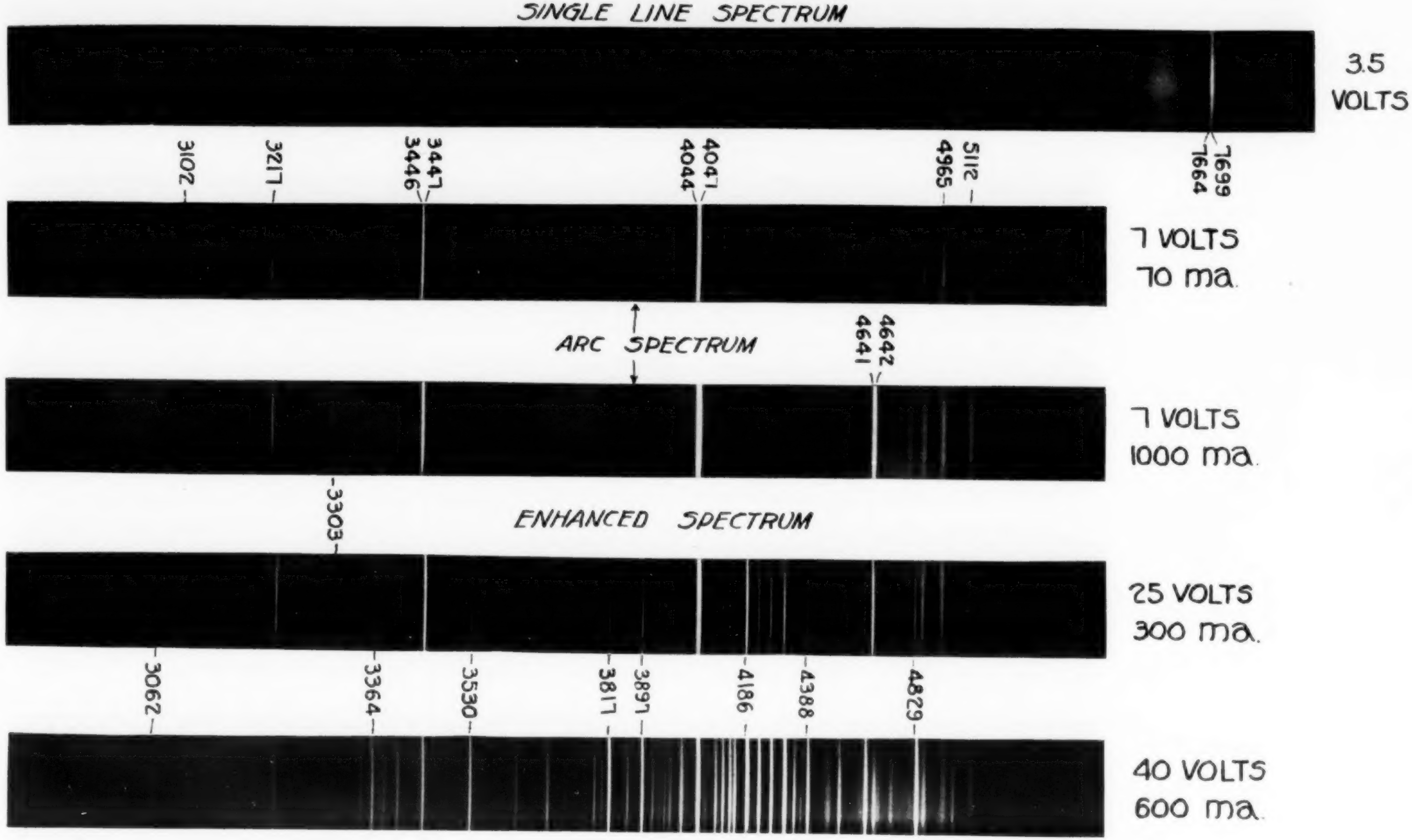
It is seen from this table that the critical voltage for the excitation of the enhanced spectrum of sodium is about 30 volts, in the neighborhood of the values for *La* and *La*.

[1] Schillinger, *Sitzungsberichte der Kais. Akad. d. Wiss.*, **118**, 605, 1909.

[2] Datta, *Proc. Roy. Soc.* (A), **99**, 69, 1921.

[3] *Annalen der Physik*, **31**, 233, 1910.

PLATE III



SPECTRA OF POTASSIUM

POTASSIUM

Plate III shows the three stages in the development of the potassium spectrum. Table II contains the wave-lengths and estimated intensities of the lines at various exciting voltages from 3.5 to 40 volts. From the resonance potential, 1.6 volts, to the ionization potential, 4.3 volts, the single-line spectrum is emitted, consisting of the pair λ 7699 and λ 7664. Above the ionization potential the complete arc spectrum is excited while in the neighborhood of 20 volts (viz., near $M\alpha$ and Ma) the enhanced lines are prominent. A wider slit was used for this series of exposures than for sodium and the results are therefore more favorable for reproduction although less advantageous for precise measurement of wave-lengths. The values for the wave-lengths in Table II are compiled from the measurements of Shillinger and Datta. Only the stronger spark lines are listed, as these serve sufficiently to show how the intensity of the enhanced spectrum rapidly increases when the critical voltage is exceeded. A great many faint spark lines were observed in addition to those given by Shillinger, a fact also noted by McLennan,[1] but their wave-lengths are not known with sufficient accuracy to justify publication.

While the main issue of the present paper concerns the excitation of enhanced spectra, the second and third spectrograms of Plate III, showing the arc spectrum of potassium, are of considerable interest. Both of these exposures were made at 7 volts, the only difference being that an ionization current of 70 milliamperes was employed for the upper spectrogram and 1000 milliamperes for the lower. In the latter case the pair λ 4642 and λ 4641 appears as one of the most prominent lines in the spectrum. This pair has the notation $\nu = 1s - 3d$, representing an interorbital transition where the change in azimuthal quantum number is according to Sommerfeld two units.[2] It accordingly contradicts the Bohr or Rubinowicz principles of selection as applied by Sommerfeld to spectra of the non-hydrogenic type While the line

[1] *Proc. Roy. Soc.* (A), **100**, 182, 1921.

[2] In a paper by Foote, Mohler, and Meggers, to appear in the *Philosophical Magazine*, this and other exceptions to the principle of selection are considered in more detail.

TABLE II

Spectra of Potassium

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
2992.21	....	....	....	1 —	1 —	....	$1s-8p_{1,2}$
3034.8	....	1	1	1	1	....	$1s-7p_{1,2}$
3062.44	3	2	2 —	....	....	....	....
3102.03 02.25	1	3	3	2	2	....	$1s-6p_1$ $1s-6p_2$
05.31	2	1	1	....	....	....	....
29.39	1	1 —	1 —	....	....	....	....
69.89	1	1	1 —	....	....	....	....
87.91	2	1	1	....	....	....	....
3190.42	2	1	1	....	....	....	....
3201.96	1	1 —	....	....	....	....	....
17.01 17.50	4	5	5	5	5	....	$1s-5p_1$ $1s-5p_2$
20.80	1	1	....	....	....	....	....
3290.93	2	2 —	1	....	....	....	....
3312.63	2	1	1 —	....	....	....	....
45.7	5	4	2	....	....	....	....
58.33	1	1	....	....	....	....	....
60.16	1	1	....	....	....	....	....
63.25 64.56	6	5	3	....	....	....	....
73.84	2	2	1	....	....	....	....
80.97	2	2	1	....	....	....	....
3385.24	2	2	1	....	....	....	....
3404.57	5	4	2	....	....	....	....
22.26	1	1	1 —	....	....	....	....
33.61	1	1 —	....	....	....	....	....
40.36	2	1	1 —	....	....	....	....
46.722 3447.701	5	6	8	8	8	....	$1s-4p_1$ $1s-4p_2$
3530.83	5	5	3	1	....	....	....
3609.26	2	1	1 —	....	....	....	....
18.43	3	2	1	....	....	....	....
68.91	1	....	....	....	....	....	....
76.01	2	1	....	....	....	....	....
3681.52	4	3	2	....	....	....	....
3716.51	1	1	....	....	....	....	....
21.93	1	1	....	....	....	....	....
39.13	1	1 —	....	....	....	....	....
44.62	1 —	1 —	....	....	....	....	....
67.37	3	2	1	....	....	....	....
3783.19	4	3	2	1 —	....	....	....
3800.60	3	2	1	....	....	....	....
17.99	6	5	3	1	....	....	....
61.87	4	2	1	....	....	....	....

TABLE II—*Continued*

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
73.78	1	1	....	....	....	....	....
3878.31	1	....	....	....	....	....	....
3897.86	7	5	3	2	....	....	....
3923.65	3	2	1−	....	....	....	....
42.86	3	2	1	....	....	....	....
55.27	3	2	1	....	....	....	....
66.75	2	1	1	....	....	....	....
72.61	2	1	....	....	....	....	....
3995.08	2	1	....	....	....	....	....
4001.20	5	3	2	1	....	....	....
12.07	1	....	....	....	....	....	....
25.20	1	....	....	....	....	....	....
44.140	6	7	8	8	8	....	$1s-3p_1$
47.201	4	5	6	6	6	....	$1s-3p_2$
57.95	3	1	....	....	....	....	....
4114.96	5	4	2	1−	....	....	....
34.72	5	4	2	1	....	....	....
49.24	5	4	2	1	....	....	....
4186.23	9	8	4	2	....	....	....
4210.21	2	1	1	....	....	....	....
22.99	5	4	2	1	....	....	....
25.60	5	4	2	1	....	....	....
4263.32	6	5	2	1	....	....	....
4305.01	5	4	2	1	....	....	....
09.08	7	6	4	2−	....	....	....
4388.13	5	4	2	1	....	....	....
4423.72	1	....	....	....	....	....	....
4466.66	2	....	....	....	....	....	....
4505.39	4	2	1	....	....	....	....
4608.31	4	2	....	....	....	....	....
41.585 } 42.172 }	2	4	5	4	8	....	{ $1s-3d'$ { $1s-3d$
4659.58	1	....	....	....	....	....	....
4741.6	....	....	....	....	1−	....	$2p_2-10s$
45.58	....	1	1	1	1	....	$2p_2-11d$
54.5	....	....	....	....	1−	....	$2p_1-10s$
59.31	....	1	1	1	1	....	$2p_1-11d$
86.89	....	1	1	1	1	....	$2p_2-\ 9s$
4791.08	....	1	1	1	2	....	$2p_2-10d$
4800.16	....	1	....	1	1	....	$2p_1-\ 9s$
05.19	....	1	1	1	2	....	$2p_1-10d$
28.99	5	3	2	....	....	....	....
49.88	1−	1	1	2	2	....	$2p_2-\ 8s$
56.03	1	2	2	3	3	....	$2p_2-\ 9d$
63.61	1−	1	1	2	2	....	$2p_1-\ 8s$

TABLE II—*Continued*

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
4869.70.....	1	2	2	3	3	....	$2p_1-9d$
4941.964....	1	2	2	2	2	....	$2p_2-7s$
43.02.....	1	....	....	....	....	....	....
50.816....	2	3	3	3	3	....	$2p_2-8d$
56.043....	1	2	2	2	2	....	$2p_1-7s$
4965.038....	2	3	3	3	3	....	$2p_1-8d$
5005.34.....	2	....	....	....	....	....	....
55.78.....	1	....	....	....	....	....	....
84.212....	2−	3−	3−	3−	4−	....	$2p_2-6s$
97.144....	2	3	3	3	4	....	$2p_2-7d$
5099.180....	2−	3−	3−	3−	4−	....	$2p_1-6s$
5112.204....	2	3	3	3	4	....	$2p_1-7d$
5323.228....	2−	3−	3−	4−	4−	....	$2p_2-5s$
39.670....	2−	3−	3−	4−	4−	....	$2p_1-5s$
42.974....	2	3	3	4	4	....	$2p_2-6d$
5359.521....	2	3	3	4	4	....	$2p_1-6d$
....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....
7664.94.....	....	....	....	....	....	6	$1s-2p_1$
7699.01.....	....	....	....	....	....	4	$1s-2p_2$

DATA FOR TABLE II

Exposure Number	Volts	Milliamperes	Time of Exposure
2.................	40	500–600	10 min.
7.................	25	0–300	32 min.
5.................	20	100–700	15 min.
8.................	16	0–500	20 min.
18.................	6	1200	15 min.
26.................	3.5	12	7 hours

is well known, its excitation has been thought to be occasioned by the presence of the electrostatic field—an incipient Stark effect.[1] However, in the present case, with the special form of discharge tube employed, the radiation was observed only in the space shielded from the applied electrostatic field—itself only 7 volts. On referring to Table I and Plate II, it will be noted that the corresponding line λ 3427 appears in sodium. Its intensity increases very rapidly as the ionization current is increased, independently of the voltage, provided this exceeds the ionization potential.

[1] Sommerfeld, *Annalen der Physik*, **63**, 249, 1921.

The importance of this result has been pointed out in the paper referred to above, in which we suggest that the explanation may involve a reconsideration of the method whereby single azimuthal quantum numbers have been assigned to each of the s, p, b, and d terms.

The following table summarizes the values of the critical voltages characterizing the three-stage development in the spectra of sodium and potassium.

Element	Volts	Notation	Spectrum
Sodium...........	2.09	$1s-2p$	Single pair
	5.11	$1s$	Arc lines
	35-30	....	Enhanced lines and L-radiation, $L\alpha=35$, $L\alpha=30$
Potassium.........	1.60	$1s-2p$	Single pair
	4.32	$1s$	Arc lines
	23, 20	....	Enhanced lines and M-radiation

Highest frequency in enhanced spectrum: sodium = 14 volts; potassium = 11 volts. Total work of double ionization: sodium = 49 volts; potassium = 34 volts.

Bureau of Standards
December 1, 1921

THE VACUUM-SPARK SPECTRA OF THE METALS[1]

By EDNA CARTER

ABSTRACT

Vacuum-spark spectrograms of Ca, Mg, Cd, Ti and Fe were obtained *with comparison arc and spark spectra,* on portrait films, using for the most part a 20-cm concave grating spectrograph. The Fe spectrum was also photographed with a dyed film from λ 4000 to λ 6600 A. The new spectra show no striking new characteristics. In the case of Ca and also of Mg (see Plate IV), the vacuum-spark spectrum is practically identical with the spark spectrum in air; in the case of Cd (see Plate V), the arc lines are somewhat more intense; with Ti this tendency to strengthen the arc lines is even greater, and the vacuum-spark spectrum of Fe is more like the arc spectrum than the spark. In general these spectra resemble the luminescence spectra produced by cathode-ray bombardment in a high vacuum, and it is probable that the conditions of emission are very similar in the two sources.

I. INTRODUCTION

The production of a brilliant spark between electrodes very close together in a high vacuum was observed by Rowland[2] and also by Wood.[3] The nature of the discharge was studied by Loving.[4] He concludes in regard to the spectrum produced that it is characteristic of the anode, the cathode having no effect, and that it is not analogous to either the spark or the arc. The spectra which he obtained were not sufficiently intense for an exact determination of their character.

Millikan[5] developed independently a method of producing this discharge; and, by using it as a source in a vacuum spectrograph, succeeded with his co-workers in extending the spectra of various elements into the extreme ultra-violet.

It was of interest to see whether a careful study of this source would disclose any peculiarities of metallic spectra in the usual photographic region. A comparison of the vacuum-spark spectra with the arc and the spark spectra of typical metals was therefore undertaken.

[1] *Contributions from the Mount Wilson Observatory,* No. 219.

[2] Rowland, *Physical Papers,* p. 574.

[3] *Physical Review,* **5**, 1, 1897.

[4] *Astrophysical Journal,* **22**, 285, 1905.

[5] *Ibid.,* **52**, 47, 286, 1920; **53**, 150, 1921; *Physical Review,* **12**, 168, 1918. *Science,* **19**, 138, 1919.

2. EXPERIMENTAL METHOD

The vacuum sparks were produced in a glass bulb about 12 cm in diameter provided with a quartz window, which in the case of the more easily vaporized metals was removed to a distance of 30 cm on account of the sputtering. The iron electrodes consisted of cylinders about one cm long and one cm in diameter, tapering to blunt points. These cylinders were screwed on the ends of aluminum rods, which projected to the outside through side tubes and were sealed in with sealing wax. The iron electrode was retained as cathode throughout the experiments, while anodes of the other metals were made by wedging a piece of the metal into a hole in the end of the aluminum rod. The distance between the electrodes was about 1.5 mm in the case of iron, but it amounted to 3 or 4 mm with some of the other metals, which were largely vaporized during the exposure.

Evacuation was effected with a Langmuir diffusion pump, the mercury vapor being kept away from the discharge by a trap immersed in solid CO_2 and ether.

The discharge was produced with a large induction coil used with an electrolytic interrupter. A current of from 15 to 20 amperes was passed through the primary. To secure the necessary capacity across the secondary, four parallel plate condensers, consisting of copper plates about 56 by 61 cm separated by 5-mm glass plates, were connected in series. The whole was immersed in oil to prevent leakage.

About eight hours were required to obtain a spectrogram of iron, because of the tendency of the discharge to pass over into an ordinary gas discharge, with the liberation of gas from the electrodes, and because it was necessary to keep the sealing wax joints cool. The total duration of the spark discharge was estimated at about thirty-five minutes for iron and not more than ten minutes in the case of cadmium. The iron required a much higher potential gradient between the electrodes for the production of the spark than the other metals.

A spectrograph with concave grating of 20-cm radius and ordinary portrait films were used in obtaining most of the spectrograms. The iron spectrum was photographed in the region 4000–6600 A

by using an aesculin filter and films sensitized with a Wallace three-dye solution. Fairly rich spectra of iron and titanium were obtained also with a one-meter concave grating spectrograph.

The vacuum-spark spectra of magnesium and cadmium with comparison arc and spark spectra are reproduced in Plates IV and V.

3. DISCUSSION OF RESULTS

The vacuum-spark spectra of calcium and magnesium are practically identical with the spark spectra in air.

The cadmium vacuum-spark spectrum retains arc lines in somewhat greater relative intensity than they possess in the ordinary spark. The line λ 3261 is an example. This line came out also with marked intensity in the luminescence spectrum produced by cathode-ray bombardment.[1]

Titanium shows an even greater tendency to retain the arc lines. The spectrum is plainly an enhanced one, however, as is evidenced by the brightness of the lines $\lambda\lambda$ 4164 and 4172, for example, the ratios of whose intensities in the spark and the arc, as given by Exner and Haschek, are 20:2 and 15:1, respectively.

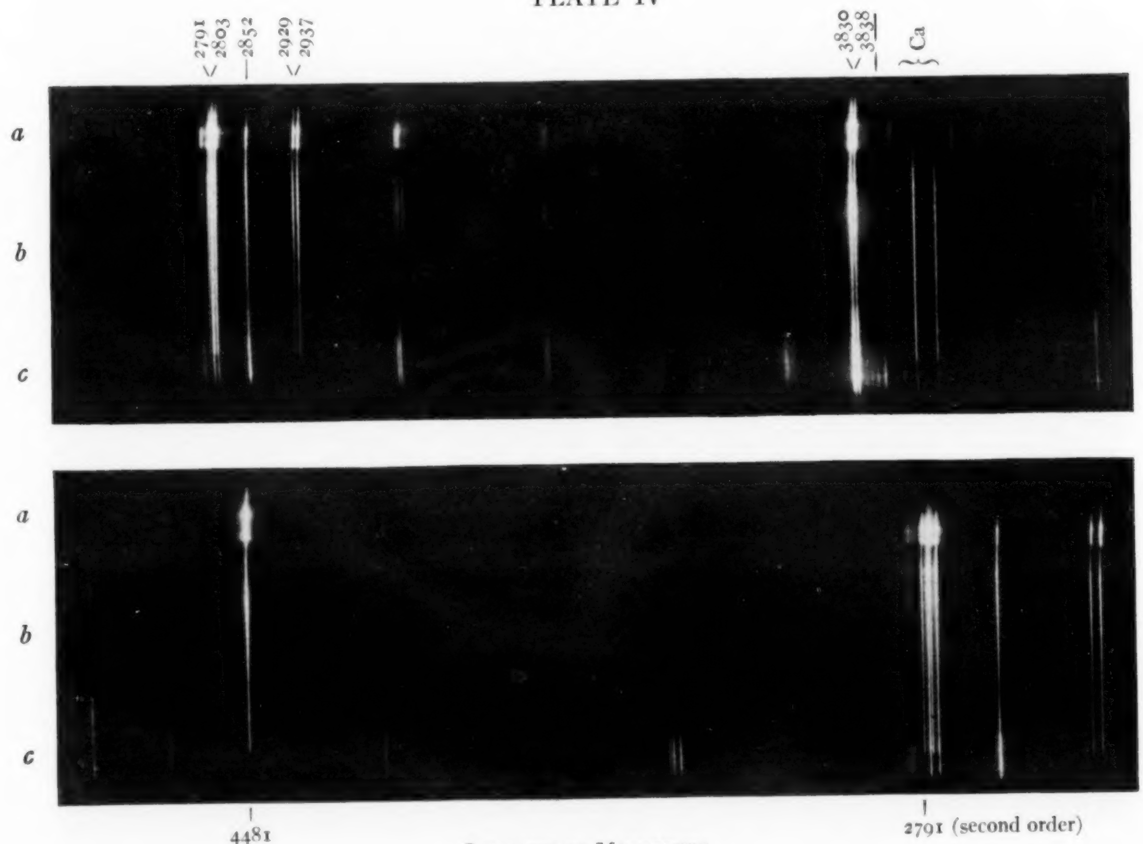
The vacuum-spark spectrum of iron, on the other hand, seems to resemble more closely the arc spectrum. The low-temperature lines found in the arc are missing, and the lines which are stronger at the poles and in the core of the arc are relatively more intense in this source. Even in this spectrum a close examination reveals the presence of the enhanced lines, such as $\lambda\lambda$ 4508, 4515, 5018, and 5316. Hα comes out very strongly and sharply in the photographs of the iron spectrum made in this region.

The varying tendency exhibited by these different metals toward the production of enhanced lines suggests a resemblance of the vacuum-spark spectra to the luminescence spectra produced by cathode-ray bombardment in a high vacuum. It is probable that the conditions of emission are very similar in the two sources. The difficulties in the way of obtaining rich spectra are considerably less in the case of the vacuum-spark.

MOUNT WILSON OBSERVATORY
September 1921

[1] *Astrophysical Journal*, **44**, 303, 1916; **49**, 224, 1919.

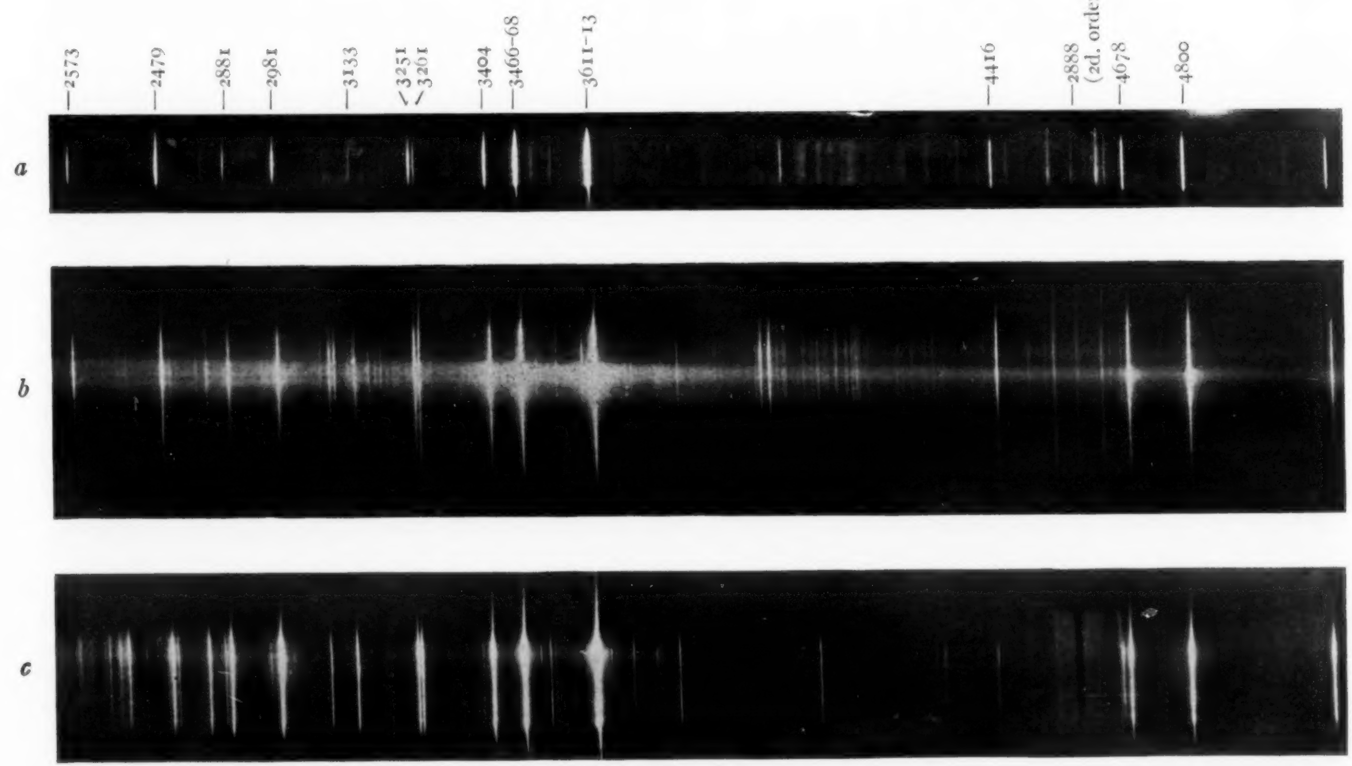
PLATE IV



SPECTRUM OF MAGNESIUM

a Spark in air
b Vacuum-spark
c Arc

PLATE V



SPECTRUM OF CADMIUM

a Spark in air
b Vacuum-spark
c Arc

THE MASSES AND DENSITIES OF THE STARS[1]

By FREDERICK H. SEARES

ABSTRACT

Mean magnitudes and masses of visual binaries and dwarf stars of the different spectral types.—A statistical comparison of the absolute magnitudes (M_b) of 105 visual binaries, the hypothetical magnitudes (M_c) of these and 400 other visual binaries (computed by Jackson and Furner, assuming the combined mass, $\mu = \mu_1 + \mu_2$, of each binary to be 2), and the absolute magnitudes (M_s) of 1152 single stars (including some binaries) by Kapteyn and Adams, gives the mean difference $(\overline{M}_b - \overline{M}_c) = (\overline{M}_s - \overline{M}_c) - 0.3 \pm 0.1$ for each type; that is, $\overline{M}_b = \overline{M}_s - 0.3 \pm 0.1$. This indicates that the selection used is sufficiently homogeneous, also that the variation of the mean magnitude of binary star components with spectral type is the same as that of single stars. The values of $\overline{M}_s$ (Table IV and Fig. 1) for dwarf stars of types B0, A0, F0, G0, K0, and Ma are -1.60, $+0.70$, 2.40, 4.35, 5.90, and 9.80 respectively. The geometric mean mass for each spectral type of visual binary was next computed from the formula $\log \mu = \log 2 - 0.6\ (\overline{M}_s - 0.3 - \overline{M}_c)$. The values of μ, slightly corrected for the effect of selection as determined by a study of the binaries of known parallax, give, when multiplied by $\frac{1}{2}$ to reduce to single stars, a first approximation to the geometric mean masses of single stars along the dwarf branch of Russell's diagram, the values for types B0, A0, F0, G0, K0, and Ma, coming out 10, 6.0, 2.5, 1.0, 0.68, and 0.59, respectively, in terms of the sun's mass (Table IV). The data for visual binaries of known parallax indicate that the distribution of $\log \mu$ is approximately Gaussian and that the *dispersion in mass* is small, probably half the stars of each type having masses between 0.7 and 1.5 times the mean (Table VIII).

Mean velocity and energy of dwarf stars of different spectral types.—The mean square space-velocity corresponding to the mean magnitude was determined from the available data for each type, and these values were multiplied by the corresponding mean masses. The mean energy for each type so obtained is practically the same for all types from A0 to Ma, in spite of a 10 to 1 variation in mass (Table IX). The logarithm of the *mean kinetic energy constant* is 3.27 ± 0.036 (average deviation per type) in terms of sun's mass and km/sec., corresponding to 3.5×10^{46} ergs. The mean radial energy is also approximately constant for all except the B-type stars, which seem to be abnormal in this as in other respects. Whatever the cause may be, the distribution of energy among these stars seems to approximate equipartition.

Masses of stars as functions of magnitude and spectral type were then computed from the values of the mean velocity for each magnitude and type, assuming the mean energy independent of magnitude as well as of type (Table X). In Figure 2, curves are given showing M as a function of type for various constant values of the mass. To get more reliable data for the giants, the masses of 28 Cepheids were computed from the period, magnitude, and surface brightness of each, using a formula based on the pulsation theory (Table XVI), and Figure 2 was revised to agree with these values, giving Figure 3. If the masses computed in accordance with Eddington's theory of Cepheids (Table XIX) are adopted, a further slight revision will be necessary. The constant mass lines would be less irregular if bolometric magnitudes were used instead of visual. For the Cepheids, the probable *dispersion in mass* for a given M and type is only about 20 per cent. This result taken with the data for visual binaries indicates that in general the luminosity of a star is very closely determined by its spectral type (temperature) and its mass.

Effective temperature, color index, and surface brightness of stars of different spectral types.—The effective or black-body temperatures of the giants as determined from

[1] *Contributions from the Mount Wilson Observatory*, No. 226.

Wilsing's spectrophotometric measures of c/T for 199 stars, vary from 2890° for Mc to 10,500° for Bo, while those of the dwarfs as determined from Mount Wilson color indices (provisional values, hitherto unpublished, also given for the giants) reduced to the same system, vary from 3330° for Ma to 6080° for F5 (Table XII). The difference between dwarfs and giants of the same type is too great to be neglected, the later-type dwarfs being considerably hotter. The surface brightness j as a function of temperature may be computed (1) from the difference between visual and bolometric magnitudes as determined by Eddington ($j=M-M_B-10 \log T+\text{const.}$) and (2) from Hertzsprung's formula derived from Planck's radiation law ($j=2.3\ (c/T)^{0.93}+\text{const.}$), both formulae giving practically the same results (Table XI). Combining these values with the temperatures of the stars of various types, we get the surface brightness as a function of spectral type, the results varying from 4.81 for $T=2890°$ to -2.28 for $T=10{,}500°$, expressed in magnitudes, sun$=0$ (Table XII). As a check on these values, the surface brightness of three stars, α Orionis, α Boötis, and α Scorpii, whose diameters have been determined directly by Pease, were computed and found to agree with the interpolated values within $\pm 0^M2$ in each case (Table XIII).

Diameter and density of stars as a function of magnitude and spectral type were readily computed from the surface brightness, absolute magnitude, and mass (Table XIV). For dwarfs the diameter decreases from 6.8 to 0.54 (sun$=1$) and the density increases from 0.045 to 5.4 (water$=1$) as the type changes from Bo to Ma, while for giants of zero magnitude the corresponding changes of diameter and density are from 3.2 to 66 and from about 0.3 to 10^{-5}. When M is plotted as a function of type for various constant values of the density (Fig. 2), the equidensity lines are straight and nearly parallel and quite regularly spaced, so that the equation: $\log \rho=-1.22\,S+0.57\,M+1.11$, where S is the type number beginning with 0 for Bo, represents the results quite closely for a wide range of density. For dwarfs of types B to F, the values agree satisfactorily with the values found by Shapley for the densities of eclipsing variables of these types. The giant part of the diagram, however, is less reliable, and is later modified in Figure 3 to agree with the values for 28 Cepheids (Table XVI) computed from their periods by the use of the theoretical formula: $\log \rho=-2 \log P+\text{const.}$ Adoption of Eddington's formula for Cepheids would lead to only a slight further modification (see Table XVII).

Relation of the observed variation of mean mass and of M with spectral type to the problem of stellar evolution.—For the dwarf branch, the decrease in mass from 10 for Bo to 0.6 for Ma may be partly, and perhaps wholly, accounted for by selection, the earlier and brighter stars being observed throughout a larger volume; further, only massive stars can attain high temperatures. The possibility that *mass may decrease with loss of energy by radiation* is also suggested, a possibility which agrees with the theory of relativity and does not necessarily conflict with Newtonian mechanics. For the giants, the observed change in M with spectral type (mass constant) is difficult to account for. Probable changes in mean atomic weight are insufficient. Changes in the opacity are doubtless involved, and likely also the source of energy, which almost certainly is not wholly gravitational and probably not independent of the time; for instance, the probable degree of ionization involves an amount of energy greater than seems to be available from gravitational forces. The intersection of equal-mass lines with the dwarf branch at large angles (Figs. 2, 3), however, does not conflict with the gravitational contraction hypothesis.

Eddington's radiative equilibrium theory of giants.—Although this involves the assumption that the ratio of the energy flow to the gravitational acceleration is constant throughout a star, whereas the ratio probably decreases considerably from center to outside surface, the calculated variation of magnitude with mass for a given spectral type is in fair agreement with observed values (Table XXII), but for a given mass the absolute magnitude is not constant as required by theory. *For Cepheids,* the theory gives the formula: $\log \rho=-2 \log P-2 \log (\gamma\alpha)^{\frac{1}{2}}+\text{const.}$, and this combined with the empirical relation, mass is proportional to radius, which Eddington's results suggest as a *condition for Cepheid variation,* gives: $M=-5 \log P+j-5 \log (\gamma\alpha)^{\frac{1}{2}}+\text{const.}$, which agrees with the observed variation of M with P for $P>2$ days, quite closely (Table XVIII and Fig. 5). This shows the consistency of Eddington's theory of Cepheids with his theory of radiative equilibrium

and tends to confirm the relation of mass to radius, which is equivalent to the condition that the average heat content per unit mass is constant. Various consequences of this condition are given. The theoretical values of mass and density given by the pulsation theory are close to those indicated by Figure 3 (Table XVII).

Ionization and mean atomic mass inside stars.—Theoretical computations indicate that the temperature and pressure at various points inside a star change in such a way as to keep the degree of ionization of the same order throughout (Table XX); hence the mean atomic mass is also about constant. When we consider giant stars of constant mass but of increasing temperature, we find that while the densities change very greatly, the high degree of ionization in the earliest stages is maintained with but little change (Table XXI), so that the mean atomic mass should decrease but slowly as development proceeds.

Comparison of spectroscopic parallaxes of visual binaries with the hypothetical parallaxes of Jackson and Turner shows systematic differences for their Table II (Table III).

Information on the masses of stars has its origin in what can be learned of the masses of binary systems, spectroscopic and visual. Any discussion of spectroscopic binaries from the standpoint of mass is complicated by lack of definite information on the inclination of orbital planes, while in the case of visual binaries the lack is that of accurate knowledge of stellar distances. Approached from either standpoint, the problem, with rare exceptions, must be treated by statistical methods whose application presupposes an abundance of data. Any solution is therefore involved in some uncertainty, which is not lessened by the fact that a transfer of results thus obtained to the stars at large involves assumptions of comparability: we do not know, for example, how the mass of an average binary is related to that of an average single star of the same type and luminosity, or even that the relation is the same for all classes of stars. The uncertainty, however, is probably not great; at any rate, we can proceed only by making such assumptions.

The following discussion is based on the visual binaries. These lead to values of the masses of stars along the dwarf branch of the Russell diagram, including those of B-type spectra. The average mass decreases continuously with advancing spectral type. The combination of these results with the space velocities of large numbers of stars recently published by Adams, Strömberg, and Joy shows that the mean kinetic energy of translation of the A's and the dwarfs is approximately constant. The principle of equipartition is then used to derive values of the masses of stars of other luminosities. The distribution curves thus obtained for mass as a function of spectral type and absolute magnitude are a first approximation

in so far as they concern stars of highest luminosity, but in the vicinity of the dwarf branch they should be reliable within a moderate percentage.

Valuable information on density has at various times been obtained from the eclipsing variables, but it has been difficult to correlate these results with other stellar characteristics, except loosely, because here again the distances, and hence the luminosities, of individual stars are inadequately known. Density, however, depends in a simple way on mass, total luminosity, and surface brightness. The surface brightness can be computed from radiation formulae, the results being checked in part by their agreement with Pease's measures of angular diameters. Hence with mass once known as a function of spectral type and absolute magnitude, the values of the density follow as a matter of course. The results thus found for mass and density are subject to control and readjustment with the aid of the Cepheid variables, whose relative densities and masses can be calculated directly from their periods, spectra, and surface brightness.

The lines of equal mass for the giant stars in Russell's diagram are irregular curves, considerably inclined to the axis of zero absolute magnitude. If therefore a giant star in its evolutionary development follows an equal-mass line, its luminosity does not remain constant during the giant stage as has often been supposed. The curves of equal density are approximately linear and parallel to the line of maximum frequency for the dwarfs. The correlation of mass with spectral type along the dwarf branch can be explained, in part at least, as a consequence of selection, although other factors may also enter. The concluding sections of the paper discuss the relation of these results to certain phases of Eddington's theory.

I. THE METHOD

The mean masses of the visual binaries of different spectral types may be found by a statistical comparison of the absolute magnitudes corresponding to their hypothetical parallaxes with the absolute magnitudes of stars of known parallax. The relation

$$\mu=\mu_1+\mu_2=\frac{a^3}{\pi^3 P^2} \tag{1}$$

leads at once to

$$\log \mu = k - 0.6M \tag{2}$$

where

$$k = 3 \log a - 2 \log P + 0.6m + 3$$

and where the absolute magnitude is defined by

$$M = m + 5 + 5 \log \pi . \tag{3}$$

For a hundred systems of known orbital elements Jackson and Furner[1] have used equation (1) to calculate the hypothetical parallax π_c corresponding to an assumed mass $\mu = 2$. For about 450 systems of unknown elements they have calculated hypothetical parallaxes from a formula depending on changes in position angle and distance, the assumption for the mass again being $\mu = 2$. The absolute magnitudes M_c corresponding to the hypothetical parallaxes for the two groups of stars are given in Tables I and II of their paper.

For any system of known elements, by (2),

$$0.30 = k - 0.6\, M_c$$

and by combination with (2) itself,

$$\log \mu = 0.30 - 0.6\, \Delta M \tag{4}$$

where $\Delta M = M - M_c$. Hence for a group of stars[2]

$$\left.\begin{aligned} \overline{\log \mu} = \log \underline{\mu} &= 0.30 - 0.6\, \overline{\Delta M} \\ \overline{\Delta M} &= \overline{M} - \overline{M}_c \end{aligned}\right\} \tag{5}$$

in which $\underline{\mu}$ represents the geometrical mean mass.

Equations (4) and (5) are the basis of the present discussion. Since the hypothetical parallaxes derived from changes in position angle and distance involve the mean inclination of the orbit-planes, the individual values of π_c thus found will differ from those obtained with the rigorous relation (1). Equation (4) therefore holds for individual stars only in the case of systems of known elements. Formulae (5), however, are valid for any representative group of

[1] *Monthly Notices*, **81**, 2, 1920.

[2] Upon re-reading Russell's address, "Relations between the Spectra and Other Characteristics of the Stars," *Publications of the American Astronomical Society*, **3**, 22; *Popular Astronomy*, **22**, 275, 331, 1914, I find that in principle the method is essentially one used by him.

stars, whether their elements are known or not, and can therefore be applied to objects in both the lists of Jackson and Furner.

To determine the mass from equations (4) and (5), the values of M, or at least of $\overline{M}$, must be known for the various spectral types. For a certain number of systems direct determinations of distance are available. Hence, M is known, and the use of (4) gives at once values of μ for the individual stars. These results will be affected by large uncertainties, but the means for groups of stars should be accurate within a small percentage. For most of the binaries, however, only the hypothetical absolute magnitude M_c is known; no other approximation for M is available, and we must proceed indirectly. This we might do by using values of $\overline{M}$ derived from single stars of known parallax whose selection with respect to the stars as a whole presents the same characteristics as those of the binaries themselves. Similarity of selection in the two lists of stars—binaries and single stars—is, however, difficult to attain in practice. It will be sufficient if we can find a list of single stars whose differences in selection as compared with the binaries are the same for all spectral types.

Let $\overline{M}_s$ represent the mean absolute magnitude for a given spectral type of the single stars of known parallax, and let $\overline{M}$ be the corresponding magnitude of the binaries of the same type. We may then write

$$\overline{M}=\overline{M}_s+\delta M \tag{6}$$

where δM indicates the influence of difference in selection. If this difference is the same for all types, δM will be constant, and we shall then speak of the selection as homogeneous.

The second of (5) then becomes

$$\overline{\Delta M}=\overline{M}_s-\overline{M}_c+\delta M=\overline{\Delta M}_s+\delta M \tag{7}$$

where $\overline{M}_c$ is the mean hypothetical absolute magnitude of the binaries of the spectrum under consideration. The value of $\overline{\Delta M}_s=\overline{M}_s-\overline{M}_c$ is found by comparing the two lists of stars for which it is presupposed that the selection for different spectral types is homogeneous, while the constant δM is given by

$$\delta M=\overline{\Delta M}-\overline{\Delta M}_s=\overline{M}-\overline{M}_c-\overline{\Delta M}_s \tag{8}$$

applied to the systems of independently determined parallax referred to above. M represents here the magnitude corresponding to the measured parallax of a binary whose hypothetical absolute magnitude is M_c. In other words, we derive δM by (8) from the hundred or more binaries of measured parallax, calculate $\overline{\Delta M_s}$ for each type by comparing the five hundred and fifty-odd stars in the lists of Jackson and Furner with the homogeneously selected single stars of known parallax, form $\overline{\Delta M}$ by (7), and, finally, solve for μ by means of the first of equations (5). Since the question of homogeneity of selection can be put to a direct test, the only assumption underlying this procedure is that the relation of mass to luminosity is the same for both binaries and single stars, which may be accepted as plausible at least.

The values of $\overline{M}_s$ and $\overline{M}_c$, whose difference is $\overline{\Delta M}_s$, are readily found by a graphical process. Plotting the values of M_c against spectral type, we find for the binaries a frequency diagram similar to that first given by Russell. The late-type giants are not numerous, but the dwarf branch is well defined. The points of maximum frequency for the dwarfs define a curve which joins smoothly with the line of modal values of the B- and A-type stars. This curve expresses the variation of $\overline{M}_c$ with spectral type, the giants of late type being disregarded altogether for the present. For $\overline{M}_s$ a similar curve was found from the data in the list of spectroscopic parallaxes,[1] supplemented by those for about 430 helium stars whose parallaxes have been determined by Kapteyn.[2] These two curves are shown in the central part of Figure 1. Their ordinates and differences in ordinates are given in the second, third, and fourth columns of Table IV. The data for the $\overline{M}_s$ curve could have been greatly increased by including results from trigonometric parallaxes. Little would have been gained in precision, however, for, as it is, the number of stars entering into the $\overline{M}_s$ curve is more than twice that for the $\overline{M}_c$ curve.

[1] Adams, Joy, Strömberg, and Burwell, *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[2] *Mt. Wilson Contr.*, Nos. 82, 147; *Astrophysical Journal*, **40**, 43, 1914; **47**, 146, 255, 1918.

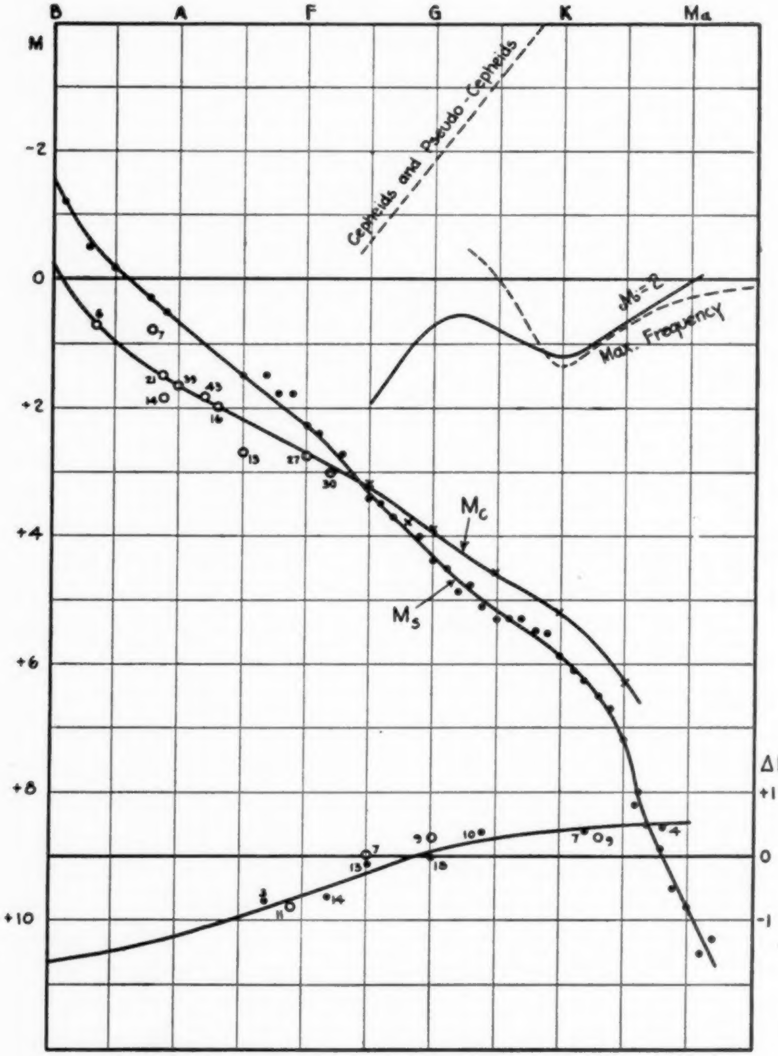


FIG. 1.—The curve M_s is the line of maximum frequency of M for the B stars of Kapteyn and the dwarfs from the list of spectroscopic parallaxes. M_c is the similar curve for the hypothetical absolute magnitudes of Jackson and Furner. The differences in the ordinates, corrected for zero point, are shown by the curve below (ΔM). The remaining curves refer to the discussion in section 16.

If we may assume that the selection for different spectral types is homogeneous, the differences in the ordinates of the two curves represent the values $\overline{\Delta M}_s$ as a function of spectrum. Irregularities of selection, or a progressive change in selection with type, will produce deformations or a tilt of the curve for $\overline{\Delta M}_s$. The curve for $\overline{\Delta M}$, formed directly from the curves for $\overline{M}_c$ and $\overline{M}_s$ and corrected for δM, is shown in the lower part of Figure 1 (the points are explained later). Since the curve is smooth, we conclude that there are no marked irregularities of selection, or at least that their influence has been minimized in drawing the curves for M_s and M_c.

2. INFLUENCE OF SELECTION ON THE STATISTICAL COMPARISON

To examine the question of selection more closely, consider the numbers of binaries and single stars of known parallax in each interval of apparent magnitude. Excluding the late-type giants and a few scattering stars at the ends of the sequence of apparent magnitudes, we find the data in the first two divisions of Table I. If the selection for the two lists were the same (not merely homogeneous), the ratios of the corresponding numbers for each type would not vary with the magnitude. Difference in selection for the binaries and single stars of any type implies a change in ratio with increasing magnitude; and, if the selection is homogeneous, the ratios for different types will show the same change. Absolute values of the ratios are of no significance, and for ease of comparison the ratios for each type have been multiplied by a constant factor which reduces that for median magnitude 6.5 to unity. Further, the numbers for the G5–K5 binaries have been combined with the mean numbers for single stars of F0–F9 and K0–K9 types.

The results in the third division of Table I indicate a satisfactory degree of homogeneity. The irregularities affecting the bright G5–K5 stars are unimportant, since the individual groups include only two or three stars each. The only deviations requiring comment are those shown by the faint G5–K5 stars and the faint B's. The latter arise from the fact that Kapteyn's lists of B stars include none fainter than the 6th magnitude. The binaries

involved are ten in number, all of the 7th magnitude or fainter, and all but one or two of types B8 or B9. Any error in the corre-

TABLE I

NUMBERS OF STARS—INFLUENCE OF SELECTION

Spectrum	Median Apparent Magnitude								Total No.
	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	
	Binaries—Jackson and Furner								
Oe5–B9	1	1	7	7	7	8	2	0	33
A0 –A9	4	3	12	32	42	25	15	1	134
F0 –G0($M>+1$)	0	6	15	36	66	78	54	2	257
G5 –K5($M>+3$)	0	1	3	4	9	24	39	1	81
Total	......	......	......	......	......	......	......	......	505
	Single Stars*—Kapteyn and Adams								
Oe5–B9	23	42	137	194	30	0	0	0	426
A0 –A9	0	7	11	11	8	3	(2)	(1)	43
F0 –F9($M>+1$)	2	14	48	105	63	50	29	3	314
G0 –G9($M>+3$)	0	5	10	35	57	54	40	8	209
K0 –K9($M>+4$)	0	1	2	14	26	43	62	12	160
Total	......	......	......	......	......	......	......	......	1,152
	Ratio—Binaries/Single Stars								
B	......	0.1	0.2	0.2	1.0	∞	∞	......	......
A	......	0.1	0.2	0.6	1.0	1.6	1.5	0.2	......
F	......	0.4	0.3	0.3	1.0	1.5	1.8	0.7	......
G5–K5	......	1	3	0.8	1.0	2.2	3.5	0.5	......
	Ratio—Jackson and Furner/Pickering								
B	......	......	7	2	1.0	0.52	0.08	0.00	......
A	......	......	3	2	1.0	0.29	0.04	0.00	......
F	......	......	3	2	1.0	0.38	0.10	0.01	......
G5–K5	......	......	3	2	1.0	0.80	0.54	0.02	......
	Ratio—Kapteyn and Adams/Pickering								
B	......	......	34	13	1.0	0.00	0.00	0.00	......
A	......	......	17	4	1.0	0.18	0.03	0.02	......
F	......	......	12	6	1.0	0.25	0.05	0.01	......
G	......	......	3	4	1.0	0.33	0.08	0.02	......
K	......	......	0.4	1.5	1.0	0.46	0.34	0.12	......

* The "single stars" include the brighter members of a few binaries.

sponding $\overline{\Delta M_s}$ must be small, for there is no pronounced irregularity in the curve. The value of $\overline{M}_s$ for Kapteyn's earlier B's

can scarcely have been affected by the limitation in apparent magnitude, for these stars are all so luminous that the limitation is of little consequence.

Among the stars fainter than $m=7.0$ there is an excess of G5–K5 binaries which represents a considerable fraction of the total number of G5–K5 stars. The corresponding $\overline{M}_c$ may be slightly too large. But, if appreciable, this effect must also be small, for as the stars actually observed in the intervals of fainter apparent magnitude are successively added, the value of $\overline{M}_c$ increases very slowly. A direct test is afforded by the binaries of measured parallax considered in a later paragraph.

The number of A stars in the spectroscopic list is small, but this does not seriously interfere with the determination of the curve of modal values because there are well-determined points for the early B's and for the F's. It does suggest, however, the advisability of an independent test of the selection of the much larger group of A's among the binaries, whose numbers can be considerably changed without greatly affecting the ratios in the middle section of Table I. To cover this point and as a further general test, both groups of stars have been compared with the counts of W. H. Pickering.[1] Using means of his values for the galactic poles and the galaxy itself (except for the B's, for which the galactic counts alone have been employed), we find the ratios in the last two sections of Table I. There is here some progression with type due to the fact that Pickering's counts include the late-type giants; but the ratios for the A-type binaries are in general agreement with those of the adjacent classes, which answers the question as to the selection of the A's. The large ratios for the bright B's in the last section of the table show merely what appears directly from the counts, namely, that Kapteyn's lists are not complete beyond the 6th magnitude. There seems also to be a defect of fainter A's in the spectroscopic list, a conclusion apparently confirmed by the fact that the points for these stars lie above the smooth $\overline{M}_s$ curve shown in Figure 1. It seems clear that none of the questions raised is serious and that we may accept the

[1] *Publications of the Astronomical Society of the Pacific*, **33**, 140, 1921.

values of $\overline{\Delta M_s}$ found in the manner described and proceed to the determination of δM.

3. ZERO POINT OF THE CURVE FOR MASS AND SPECTRUM

It will be noted that the values of $\overline{\Delta M_s}$ determine the rate of change of mass with spectral type, while the constant, δM, determines the zero point. For the calculation of δM only the spectroscopic parallaxes have been used, since these are based on a homogeneous system and are sufficiently numerous to afford an excellent determination. In forming values of $M-M_c$ for individual stars to be substituted into (8), two points must be borne in mind: First, the values of M given in the spectroscopic list correspond to the most probable values of the parallaxes, and, because of the logarithmic relation connecting absolute magnitude and parallax, are not the most probable magnitudes themselves. To obtain the latter, a small constant correction of +0.08 mag. must be applied to the tabular values.[1] Second, the values of M_c given by Jackson and Furner usually represent the total luminosity of the system. To make them comparable with values of M in the spectroscopic list, they must be reduced to the brighter component.

A summary of the results for δM is given in Table II, the two lists of Jackson and Furner being treated separately. The values of $\overline{\Delta M_s}$ required for equation (8) were interpolated from the fourth column of Table IV. The adopted result is $\delta M = -0.3$, whence equation (7) becomes

$$\overline{\Delta M} = \overline{\Delta M_s} - 0.3. \qquad (9)$$

It will be noted that there is no marked progression in the values of δM with type, which confirms, in part at least, the foregoing conclusion as to homogeneity of selection. This is also shown by the lower part of Figure 1, in which the curve is that representing equation (9), while the points correspond to the values of $\overline{M-M_c}$ in Table II. With the exception of the constant term, the curve depends wholly upon the statistical comparison, while the points are derived from values of M and M_c referring to the same star

[1] *Mt. Wilson Contr.*, No. 199, p. 15; *Astrophysical Journal*, **53**, 27, 1921.

and are uninfluenced by any possible dissimilarity in selection. The agreement of the points with the curve is, however, within the uncertainty affecting their positions.

TABLE II

ZERO POINT OF CURVE FOR $\overline{\Delta M}_s$

J and F, Table I					J and F, Table II				
Sp.	$\overline{M-M}_c$	$\overline{\Delta M}_s$	δM	No. Stars	Sp.	$\overline{M-M}_c$	$\overline{\Delta M}_s$	δM	No. Stars
A7.....	−0.7	−0.5	−0.2	3	A9....	−0.8	−0.4	−0.4	11
F2.....	−0.6	−0.2	0.4	14	F5....	0.0	0.0	0.0	7
F5.....	−0.1	0.0	0.1	13	G0....	+0.3	+0.4	−0.1	9
G0....	0.0	+0.4	0.4	18	K3....	+0.3	+0.8	−0.5	9
G4....	+0.4	+0.6	0.2	10					
K2....	+0.4	+0.7	0.3	7					
K8....	+0.5	+0.8	−0.3	4					
Mean and total......			−0.29	69	Mean and total......			−0.27	36

A further detail bearing on the question of selection is the circumstance that the constant δM arises almost wholly from the use of the tabular values of M_c for the derivation of the curve for $\overline{M}_c$. Since these generally represent total luminosity, the reduction to the brighter component appears in δM, and in the mean is nearly equal to the value found for this constant. The effect upon $\overline{\Delta M}_s$ of the obvious differences in selection shown in Table I is therefore very small. Consequently, the influence of small deviations from homogeneity in selection in passing from type to type may be regarded as altogether negligible.

4. SYSTEMATIC DIFFERENCES IN THE HYPOTHETICAL PARALLAXES

Although the two mean values of δM given in Table II are practically identical, this was not the case when the calculation was originally made, the difference then amounting to several tenths of a magnitude. This discordant result has been traced to a systematic difference between the parallaxes of the two lists of Jackson and Furner, the character of which may be seen from the comparisons shown in Table III. For systems of known elements (J and F, Table I) the agreement of the mean hypothetical paral-

laxes with the spectroscopic results is excellent; the one large difference $+0\overset{''}{.}014$ is only 7 per cent of the corresponding π_c and would be reduced to $+0\overset{''}{.}004$ by the rejection of a single star, Bu 2109.

TABLE III

COMPARISON OF SPECTROSCOPIC AND HYPOTHETICAL PARALLAXES

π_c	J and F, Table I			J and F, Table II			J and F, Table I—Table II		
	Mean π_c	$\pi-\pi_c$	No.	Mean π_c	$\pi-\pi_c$	No.	Mean π_c*	$\pi-\pi_c$	No.
$<0\overset{''}{.}020$	$0\overset{''}{.}016$	$+0\overset{''}{.}001$	7	$0\overset{''}{.}015$	$-0\overset{''}{.}002$	8	$0\overset{''}{.}015$	$-0\overset{''}{.}001$	6
0.020–0.049........	0.030	−0.001	28	0.036	−0.009	16	0.024	−0.007	12
0.050–0.099........	0.065	−0.003	22	0.064	−0.012	13	0.043	−0.012	8
>0.100	0.198	+0.014	10	0.209	−0.071	1	......	....	....

* The intervals for π_c are $\gtreqless 0\overset{''}{.}020$, $0\overset{''}{.}021$–$0\overset{''}{.}029$, and $\lesseqgtr 0\overset{''}{.}030$.

For the stars in the second list, however, the agreement for the first group alone is within the uncertainty; for the others the differences are systematic and an important percentage of the parallaxes themselves. Further, the same difference is shown by a direct comparison of the hypothetical parallaxes of twenty-six stars which occur in both the lists of Jackson and Furner. The means, arranged in three groups, are in the last division of Table III. For half of these stars no spectroscopic parallaxes are available, and to that extent the second comparison is independent of the first.

The difference, which is about 20 per cent of the parallax, corresponds to a difference of 0.4 mag. in the two values of δM and fully accounts for the discordance. For the final determination of δM, the hypothetical parallaxes from the second list were reduced to the spectroscopic system in accordance with the indications of Table III, in order that all the results may refer consistently to this system. The resulting value of δM is that given in the right-hand division of Table II.[1]

[1] I am greatly indebted to Mr. Adams for the spectroscopic parallaxes of nineteen binaries in the first list of Jackson and Furner (six stars also occur in the second list) which have been obtained since the reductions described in the text were finished. These reductions have not been revised, but I have satisfied myself that the inclusion of the new data would not change the results in section 4 or the following sections

5. PROVISIONAL MASSES—EFFECT OF SELECTION

The derivation of values of the masses now requires only the substitution of $\overline{\Delta M}$ from (9) into the first of (5). The results for log $\underline{\mu}$ are in the fifth column of Table IV. The first and last

TABLE IV

ADOPTED GEOMETRICAL MEAN MASS ($\odot = 1$), SPECTRAL TYPE AND ABSOLUTE MAGNITUDE

Sp.	M_s	M_c	$\overline{\Delta M_s}$	Provisional Log $\underline{\mu}$	Corr. for Selection	Vis. Binaries		Single Stars	
						Log $\underline{\mu}$	$\underline{\mu}$	Log $\mathfrak{M}$	$\mathfrak{M}$
B0	−1.60	(−0.25)	(−1.35)	1.29	−0.04	1.25	18	1.01	10
B5	−0.20	+1.00	1.20	1.20	− 4	1.16	14	0.92	8.3
A0	+0.70	1.65	0.95	1.05	− 3	1.02	10.5	0.78	6.0
A5	1.50	2.15	0.65	0.87	− 3	0.84	6.9	0.60	4.0
F0	2.40	2.70	−0.30	0.66	− 2	0.64	4.4	0.40	2.5
F5	3.32	3.27	+0.05	0.45	− 2	0.43	2.7	0.19	1.5
G0	4.35	3.95	0.40	0.24	− 1	0.23	1.7	9.99	1.0
G5	5.20	4.60	0.60	0.12	0	0.12	1.3	9.88	0.76
K0	5.90	5.20	0.70	0.06	+ 1	0.07	1.2	9.83	0.68
K5	7.10	6.30	0.80	0.00	+ 3	0.03	1.1	9.79	0.62
Ma	+9.80	(+8.95)	(+0.85)	9.97	+0.04	0.01	1.0	9.77	0.59

values depend upon extrapolations, and it is possible that those for B5 and A0 are also affected by considerable uncertainty. So far as the statistical comparison is concerned, the remaining values should be dependable within 5 or 10 per cent.

These results can be accepted, however, only with some reservation, for it is easily seen that the visual binaries are a selected class of stars in the characteristic of mass itself, and especially is this true of the more distant objects. Since motion is detected only with difficulty, or not at all, in case the period is very long, equation (1) shows that the selection of large masses is favored to an increasing degree as more and more distant stars are considered. For a given value of a the factor a^3/π^3 increases rapidly with increasing distance; the smaller masses are therefore excluded in succession because they correspond to periods so long that motion has not been observed. For large values of π a wide range in mass

in any essential particular. Thus the first two differences in the first division of Table III would be $+0\overset{''}{.}002$ and $-0\overset{''}{.}002$ for seventeen and thirty-seven stars, respectively. The differences in the third division would not be changed by the inclusion of the six stars also occurring in the second list.

will be included, but since there is a practicable lower limit to a, only the most massive among the more distant stars will appear in our lists.

Another and perhaps even more important form of selection arises from the fact that only the most luminous of the distant stars get into our lists; the others are too faint to be seen. This, too, excludes in succession the smaller masses with increasing distance, and is related to the correlation of mass with spectral type discussed in section 16.

The influence of distance on the selection of mass seems to be shown by the data now available. To test the matter, consider the sixty-nine binaries of known parallax[1] in the first list of Jackson and Furner. For these, the individual masses, μ, can be calculated by (4). Comparing these values with the mean mass for stars of the same spectrum, as interpolated from Table IV, we obtain a series of residuals $\Delta \log \mu = \log \mu - \log \underline{\mu}$. When grouped according to parallax, these lead to the mean systematic deviations shown in Table V. A progression with increasing parallax, in the direction

TABLE V

DEPENDENCE OF MASS ON PARALLAX

π_s	Mean Sp.	No. Stars	$\overline{\Delta \log \mu}$	Curve
0''.018	F2	10	−0.08	−0.11
.026	F6	10	−0.12	−0.08
.032	G1	10	−0.08	−0.07
.046	F7	10	+0.03	−0.03
.061	G3	10	−0.04	0.00
.102	G1	10	+0.04	+0.04
0.270	G8	9	+0.12	+0.12

anticipated, is evident, the negative sign of the differences associated with the more distant stars indicating that the corresponding

[1] The thirty-six stars of known parallax in the second list are not included because of the systematic difference in the parallaxes of the two lists. In the preparation of Table II, where only a mean result for all the stars was finally used, the systematic difference can be corrected with sufficient precision. Here, however, we are concerned with individual stars, or at most with small groups, and it is not certain that the mean correction is applicable. Then, too, the fact that the second list of Jackson and Furner is based on a statistical formula, renders these data unsuitable for the present discussion.

mean masses in Table IV are too large. The character of the progression is best seen from the smoothed values of the differences given in the last column of Table V. Considered as corrections, these differences are such as would reduce the masses to values representing the selection for $\pi = 0''.06$. Even at this moderate distance the smallest masses will still be excluded; hence some further correction, a constant, is required. This cannot be accurately specified; but in general it may be said that, if we regard the selection for the two nearest groups in Table V as approximately representative of the visual binaries as a whole, the data of this table would indicate that the mean masses of the binaries actually observed at distances corresponding to $\pi = 0''.02$ are roughly 50 per cent larger than those of a representative collection.

This result at once suggests that the variation of log $\underline{\mu}$ with spectral type found above has been influenced by selection, for in general the mean parallax of the earlier-type stars, which are all highly luminous, is less than that of the late-type dwarfs. Forming the mean parallax for each of the spectral groups in the first half of Table II, we find in fact a progression in distance as shown in Table VI. If the data of Table V may be accepted, the values of

TABLE VI

CORRECTIONS TO LOG $\underline{\mu}$

(To Eliminate Selection in Mass)

Sp.	π_s	No. Stars	Correction to log $\underline{\mu}$
A7	$(0''.133)$	3	−0.03
F2	0.051	14	−0.02
F5	0.038	13	−0.05
G0	0.055	18	−0.01
G4	0.069	10	0.00
K2	0.114	7	+0.06
K8	0.229	4	+0.11

log $\underline{\mu}$ in Table IV must be corrected as indicated in the last column of Table VI, in order to obtain the true variation of log $\underline{\mu}$ with spectral type. To make the results fully representative, the additional constant correction referred to above must also be applied. This seems to be of the order of −0.08.

Unfortunately, however, the results in Table V are not as reliable as could be wished. The individual values of the masses are affected by large uncertainties, and the number of stars in the separate groups is small. Further, the residuals $\log \mu - \log \underline{\mu}$ depend upon the ordinates of the curve for ΔM shown in Figure 1. Any error in the curve, in the nature of a tilt with respect to the axis, must therefore have introduced a progression in the residuals which only partially disappears when they are grouped according to parallax, because such a grouping, in part at least, is also a grouping with respect to spectral type. This uncertainty is probably not very serious, for Table V shows that the relative effect on the corrections to $\log \mu$, for the two extremes of distance, is the differential error in the ordinates of the ΔM curve for F2 and G8. Since these ordinates are not widely separated, the influence of incorrect slope in the curve cannot be great.

6. ADOPTED MASSES

On the basis of the evidence in Table VI, the corrections given in the sixth column of Table IV were finally adopted. These are the smoothed values of the results in Table VI, extrapolated to B0 and slightly modified in the case of type K because of the small number of these stars. The small constant correction referred to above is ignored, because of its uncertainty and relative unimportance. It merges with the uncertainty involved in the assumptions of comparability which must be made in extending the results for the masses of the binaries to the stars at large.

The differential correction, affecting the relative values of the masses as it does, is of more interest. The present determination is to be considered as only a first approximation; the results are at least in the right direction, and it is believed that they do not represent an over-correction.

We have, finally, as adopted values of the geometrical mean masses of the visual binaries, the data in the seventh and eighth columns of Table IV. These results refer mainly to dwarf stars and correspond to the mean absolute magnitudes given in the second column of the table.

Owing to the small number of late-type giants in the lists of visual binaries, it is not possible to use the statistical method for a determination of the masses of these stars. Some indication of their values is afforded, however, by the application of equation (4) to such of these objects as occur in the spectroscopic list. For 14 G and K stars, with a mean absolute magnitude of 0.0,

$$\log \underline{\mu}=0.47\pm0.16$$

whence $\underline{\mu}=3.0$ with an uncertainty of about 40 per cent.

The use of the geometrical mean makes it difficult to compare the masses thus found with other results,[1] for which the arithmetical mean has usually been employed. For a Gaussian distribution of the logarithms of the mass, which seems to be that indicated by the available data, the conversion formula would be[2]

$$\log \bar{\mu}=\log \underline{\mu}+2.53\,r^2$$

where $\bar{\mu}$ is the arithmetical mean, and r the probable dispersion, i.e., the median value of the differences $\log \mu-\log \underline{\mu}$, arranged in order without regard to sign. The value of r is difficult to determine. For the dwarf stars it is certainly small, probably not larger than 0.15 (see page 185), whence, approximately, $\bar{\mu}=1.14\,\underline{\mu}$.

Applied to the present data, this relation would lead to arithmetical mean masses a little larger than those previously found, but close comparisons are not possible. The change in mass with spectral type is of the general character of that resulting from other investigations and is certainly well established, although the variation given here is not as large as that derived by Ludendorff[3] from a discussion of spectroscopic binaries.

Strictly speaking, the present results refer only to the particular group and class of stars upon which the conclusions are based.

[1] Russell, *loc. cit.*; *Publications of the American Astronomical Association*, **3**, 327; *Popular Astronomy*, **25**, 666, 1917. Aitken, *The Binary Stars*, pp. 206 ff., 1918. Van Maanen, *Publications of the Astronomical Society of the Pacific*, **31**, 231, 1919. Ludendorff, *Astronomische Nachrichten*, **189**, 145, 1911; **211**, 105, 1920.

[2] Adams, Strömberg, and Joy, *Mt. Wilson Contr.*, No. 210, p. 17; *Astrophysical Journal*, **54**, 25, 1921. Although derived for use in another connection the formula is generally applicable.

[3] *Loc. cit.*

Their applicability to any other class of stars presupposes some assumption. Thus in passing from the combined masses of the visual binaries to those of single stars, we can make use of the average mass-ratio of the two components and associate the resulting average values for the central stars with that for the non-binaries of the same spectra; but it is by no means certain that the stars so associated are strictly comparable. For the further discussion we make this assumption. Presumably the uncertainty is not large, for we thus arrive at a mean mass for the solar-type stars which is the same as that of the sun itself. To give this conclusion force it must be added that the average deviation of the mass of a dwarf star from the mean mass for the type in question is small (see next section).

There is some evidence that the mass-ratio varies with the type.[1] Adopting, however, the mean value 0.75 for the ratio of secondary to principal component, we find for the stars at large the geometrical mean masses ($\mathcal{M}$) in the last two columns of Table IV.

7. DISPERSION IN MASS

The residuals in log μ for the sixty-nine stars discussed above throw some light on the dispersion in mass. Freeing the individual residuals from the influence of selection in mass, we find the frequencies shown in Table VII. In spite of the small numbers of

TABLE VII

DISTRIBUTION OF Δ LOG μ

Interval of Δ Log μ	No. Residuals		Total	
	+	−	Obsd.	Theor.
0.00–0.09	8	4	12	17
0.10–0.19	10	8	18	15
0.20–0.29	6	7	13	12
0.30–0.39	5	7	12	10
0.40–0.49	5	5	10	6
0.50–0.59	0	1	1	4
0.60–0.69	1	1	2	3
0.70–0.79	0	0	0	1
0.80–0.89	1	0	1	0
Sums	36	33	69	68

[1] Aitken, *The Binary Stars*, pp. 206, 216, 1918.

stars, it is seen from the second and third columns that the frequency curve is approximately symmetrical. Further, the last two columns show that, within the uncertainty of the data, the values of log μ follow a Gaussian distribution.

The probable dispersion (dwarfs F0 to M) is

$$r(\Delta \log \mu) = \pm 0.22. \tag{10}$$

This, however, is not a true measure of the dispersion in log μ, for the result has been derived with the aid of equation (4) and includes, therefore, the influence of errors in M and M_c. Those in M_c arise from the factor a^3/P^2 and are relatively small. Bearing on the accuracy of M are the data in *Contribution* No. 199, p. 15, from which for the spectral types and luminosities involved, $r(M) = \pm 0.36$ mag. By (4), the contribution to the calculated dispersion in log μ arising from errors in M is therefore ± 0.22, which is the same as the calculated value of $r(\Delta \log \mu)$ itself. Since the dispersion in the masses presumably is not zero, the adopted value of $r(M)$ is probably too large. This unfortunately makes it impossible to gain any definite information as to the dispersion, and we can only indicate, as in the second column of Table VIII,

TABLE VIII

PROBABLE DISPERSION IN MASS

$r(M)$	Probable Dispersion in Log μ	Limits for Prob. = ½
±0.35	±0.07	0.85μ–1.18μ
0.30	0.13	0.74 –1.35
0.25	0.16	0.69 –1.45
0.20	0.18	0.66 –1.51
±0.15	±0.20	0.63 –1.58

the probable values corresponding to various assumed values of $r(M)$. The last column of the table gives the limits within which it is an even chance that the mass of any visual binary will be found. Thus, if we assume that the true probable error of the spectroscopic absolute magnitude is ± 0.25, the masses of one-half of the stars should be between 0.7 and 1.5 times the mean mass.

Apparently the range in the masses of the dwarf stars from Fo on must be small. Further evidence is given on page 214. These results are in general agreement with the earlier findings of Russell and Eddington.[1]

8. EQUIPARTITION OF ENERGY

A possible analogy between the dynamical behavior of the stars and that of the molecules of a gas has frequently been mentioned.[2] Applicability of the kinetic theory would imply an equipartition of the energy of translation such that

$$\mu_1\overline{v_1^2}=\mu_2\overline{v_2^2}= \ . \ . \ . \ =\mu_n\overline{v_n^2} \tag{11}$$

where the individual values in $\overline{v_n^2}$ include all the velocities associated with a definite class of masses μ_n. In a gas this final state is brought about very quickly; in a collection of stars the period of relaxation, if the term has any meaning at all, obviously must be enormously long.

As a matter of fact, the theory for a collection of stars is much simpler than that of a gas, because the occurrence of collisions and close encounters is so rare that they may be neglected.[3] The effective agency for the transfer of energy is therefore only that arising from the attraction of the great mass of stars as a whole upon the individual members of the group. But this simplification of the theory and the consequent restriction of the mechanism for the redistribution of energy so enormously lengthens the interval required for the attainment of statistical equilibrium that Edding-

[1] *Monthly Notices*, **77**, 30, 1916; **604**, 1917. Russell, in this connection, makes the important suggestion that the real dispersion in mass may be largely concealed in a discussion based upon spectroscopic data. This would be the case if the spectral characteristics used for the determination of a star's absolute magnitude are functions of its surface brightness (temperature) and density alone. For Russell's discussion of this interesting point see the following note on page 238.

[2] See for example Poincaré, *Hypothèses Cosmogoniques*, p. 257, 1911; Halm, *Monthly Notices*, **71**, 634, 1911; Eddington, *Stellar Movements*, pp. 159, 247, 1914; Jeans, *Problems of Cosmogony and Stellar Dynamics*, pp. 224 ff., 1919.

[3] Jeans, *op. cit.*, p. 229.

ton[1] rejects altogether the analogy with a gas and proceeds on the principle that the motions of the stars do not mutually interfere.

As far as the present state of the universe is concerned, Jeans[2] concurs in this opinion, but points out that the conditions of an earlier stage in its history may have been entirely different. Following his conjecture that the system has evolved from a rotating nebula of dimensions much less than those of its present configuration, he shows that the earlier stages of development would be favorable to a rapid redistribution of energy, and that the system may have proceeded far on the way toward equipartition before its gradual expansion reduced the rate of transfer to its present negligible amount. The fact that readjustments are now inappreciable does not therefore demonstrate the absence of equipartition, which must be put to an observational test.

Although a certain type of systematic motion is not incompatible with a steady state, the phenomenon of star-streaming shows that equipartition has not been completely attained. Within the individual streams, however, there may be an approach to this state, just as in the case of two intermingling streams of gas. Further, if the stream motion is small as compared with the peculiar velocities of the stars, it will have little influence on the statistical averages in (11). For a decade or more some correlation of mass with velocity of the type hereby implied has been suspected, for it has been known that in general the most massive stars are the most luminous, and the most luminous stars have on the average the smallest velocities.[3] Now, however, we are in possession of data which admit of a more definite test.

For the B and A stars, whose extreme range in luminosity is small, we have the determinations of mean radial velocity by

[1] Eddington, *loc. cit.*, p. 254. See also *Astronomische Nachrichten, Jubiläumsnummer*, p. 9, 1921.

[2] *Op. cit.*, p. 237.

[3] Campbell, *Stellar Motions*, p. 207, 1913. Kapteyn, *Mt. Wilson Contr.*, No. 45; *Astrophysical Journal*, **31**, 258, 1910. Russell, *Observatory*, **37**, 174, 1914. Kapteyn and Adams, *Mt. Wilson Comm.*, No. 1; *Proc. Nat. Acad. of Science*, **1**, 14, 1915. Adams and Strömberg, *Mt. Wilson Contr.*, No. 131; *Astrophysical Journal*, **45**, 293, 1917.

Campbell.[1] For the remaining types, within which the differences in luminosity are much greater, we have the recent Mount Wilson results,[2] giving both radial and space velocities for each of these types as a function of absolute magnitude. Since the values of the masses in the last column of Table IV are probably representative of the stars at large, they may be combined with the mean velocities of the different types, in conformity with (11), to determine the degree to which the condition of equipartition is satisfied.

The values of the velocities have not been corrected for stream motion and thus subject the hypothesis of equipartition to a severe test. The Mount Wilson results for radial and space velocities include the K-term. For the late-type dwarfs, this is small, and negligible for the present discussion. Campbell, on the other hand, has freed his results from this term, which for the B stars is two-thirds of the mean radial velocity itself. If one prefers to regard the K-term as a real velocity, and not merely a systematic correction, its contribution to the energy can be found as indicated below.

It has been shown that the distribution of the space velocities for types F to M is not in accordance with Maxwell's law,[3] but agrees well with the assumption that the logarithms of the velocities follow a Gaussian distribution. Nevertheless, Campbell's theorem to the effect that the arithmetical mean space velocity equals twice the mean radial velocity, holds for the giants with close approximation.[4] We assume that a similar state of affairs exists for the B and A stars, and thus obtain for these types the arithmetical means of their space velocities. The square of the arithmetical mean space velocity could have been used directly for a test of equipartition; but to evaluate the influence of the K-term, considered as a velocity, it is simpler to use the mean-square space velocity.

[1] *Lick Observatory Bulletins*, **6**, 101, 1911; **7**, 19, 1912.

[2] Adams, Strömberg, and Joy, *Mt. Wilson Contr.*, No. 210; *Astrophysical Journal*, **54**, 9, 1921. These results are vital for the remainder of the discussion. Without them it could not have been undertaken.

[3] Kapteyn and Adams, *Mt. Wilson Comm.*, No. 1; *Proc. Nat. Acad. of Sciences*, **1**, 14, 1915. Adams, Strömberg, and Joy, *op. cit.*, p. 15.

[4] *Ibid.*, p. 13.

For a Gaussian distribution of the logarithms of the space velocities, the relation of the mean square velocity to the arithmetical and geometrical means is expressed by[1]

$$\left.\begin{aligned}\log \overline{v^2} &= 2 \log \bar{v} + \frac{1}{2h^2 \text{ Mod.}} = 2 \log \bar{v} + 0.148 \\ \log \overline{v^2} &= 2 \log \underline{v} + \frac{1}{h^2 \text{ Mod.}} = 2 \log \underline{v} + 0.296\end{aligned}\right\} \tag{12}$$

where h is the modulus of the distribution function. It is assumed that the value of h for types F to M given in *Contribution* No. 210 applies also to the B and A stars.

The mean radial velocities taken from Campbell's papers[2] are

B3, 6.2; B8.5, 6.7; A0, 10.3; A2, 11.4; A5, 12.6.

The corresponding mean square velocities were obtained by doubling the radial velocities, as indicated above, and substituting the results into the first of (12). For the later types the values of $\log \overline{v^2}$ were derived from the data for $\underline{v}$ given in *Contribution* No. 210. These were plotted—$\underline{v}$ against corresponding M—thus giving four curves, which refer to the approximate mean types F5, G5, K3, and Ma+. The mean absolute magnitudes of the dwarf stars of these types were then interpolated from Table IV and used as arguments to read from the curves values of $\underline{v}$ corresponding to the average dwarf of each of the four types. Graphical interpolation and an extrapolation to F0, followed by substitution into the second of (12), gave finally the required values of $\log \overline{v^2}$ for every

[1] The second equation is found by applying the mean value theorem to the frequency function

$$F(v) = \frac{h \text{ Mod.}}{\sqrt{\pi}} e^{-h^2(\log v - \log \underline{v})^2}$$

The first results from the combination of the second with

$$\log \bar{v} = \log \underline{v} + \frac{1}{4h^2 \text{ Mod.}}$$

which is given by Adams, Strömberg, and Joy, *op. cit.*, p. 17.

[2] *Lick Observatory Bulletins*, **6**, 127, 1911; **7**, 28, 1912.

half-interval of spectral type from F0 to Ma. These mean-square velocities are not affected by any smoothing from type to type, since the interpolation card was drawn *through* the points derived from the four curves giving v the function of M.

The results, including those for log $\mathcal{M}\overline{v^2}$, are in the first six columns of Table IX. With the exceptions of the B stars, the

TABLE IX

THE ENERGY CONSTANT

Sp.	M	Log $\mathcal{M}$	Log $\overline{v^2}$	Log $\mathcal{M}\overline{v^2}$	Residual	Log θ^2	Log $\mathcal{M}\theta^2$	Residual
B3	−0.6	0.95	2.34	(3.29)	(+0.28)	1.74	(2.69)	(+0.07)
B8.5	+0.4	0.81	2.40	(3.21)	(+ .36)	1.62	(2.43)	(+ .33)
A0	0.7	0.78	2.78	3.56	+ .01	2.00	2.78	− .02
A2	1.0	0.70	2.87	3.57	.00	2.10	2.80	− .04
A5	1.5	0.60	2.95	3.55	+ .02	2.19	2.79	− .03
F0	2.4	0.40	3.11	3.51	+ .06	2.36	2.76	.00
F5	3.3	0.19	3.36	3.55	+ .02	2.56	2.75	+ .01
G0	4.4	9.99	3.62	3.61	− .04	2.76	2.75	+ .01
G5	5.2	9.88	3.78	3.66	− .09	2.84	2.72	+ .04
K0	5.9	9.83	3.80	3.63	− .06	2.84	2.67	+ .09
K5	7.1	9.79	3.74	3.53	+ .04	2.92	2.71	+ .05
Ma	9.8	9.77	3.78	3.55	+0.02	3.06	2.83	−0.07
Means, average deviations				3.57	±0.036	......	2.76	±0.036

values of the energy show close accordance. For a range of 10 to 1 in the masses the average deviation in $\mathcal{M}\overline{v^2}$ is 9 per cent, and, by a moderate amount of smoothing, is easily reduced to 3 per cent. The large divergence for the B stars seems to be real. It cannot be supposed that the adopted values for their masses are in error by 50 per cent, nor does it seem permissible to assume that the velocities could by any chance be increased by the 2 or 3 km necessary to bring the energy up to the mean for the other types. It is more likely that the velocities are already too large, because of undetected spectroscopic binaries among the stars combined in the means.

The inclusion of the K-term as a velocity removes only a small part of the divergence. Thus, for any radial direction, the resultant mean-square velocity $\overline{r^2}$ would be

$$\overline{r^2}=\overline{v^2}+K^2-\overline{2vK\cos\alpha}=\overline{v^2}+K^2$$

where α is the inclination of the space velocities to the given line of sight. Extending this to all radial directions, we have for the entire group of B3 stars,

$$\overline{r^2}=(14.8)^2+(4.7)^2=241.1;\ \log \mathcal{M}\overline{v^2}=3.33,$$

an increase of only 0.04 in the logarithm of the energy.

The foregoing results depend upon the assumptions that the mean space velocities of the B and A stars may be found by doubling the mean radial velocities, and that the modulus h applies equally to all types. We may avoid these assumptions by restricting the test to the mobility in the line of sight. For this comparison the K-term has been included throughout. The mean radial velocities from Campbell's data thus become[1]

B3, 7.45; B8.5, 6.50; A0, 10.1; A2, 11.3; A5, 12.5 km/sec.

For the remaining types *Contribution* No. 210 was again used, the values being read from curves for radial velocity θ as a function of absolute magnitude as in the case of $\underline{v}$. The values of $\log \mathcal{M}\theta^2$ (eighth column, Table IX) show about the same degree of accordance as those of $\log \mathcal{M}\overline{v^2}$, the average deviations, without smoothing, again being 9 per cent. The inclusion of the K-term as a part of the radial velocity has brought the B3 stars into agreement with the mean for types A to Ma, and only the B8–B9 stars are now divergent.[2] But this improved agreement is not to be taken very seriously, and perhaps the most satisfactory point of view is that since the B stars are peculiar in many particulars, the abnormality shown by Table IX should not be looked upon as surprising,

[1] *Ibid.*, **6**, 108. The means for the B stars are those found by correcting Campbell's values V_1 by 4.7 (B0–B5) and 4.1 (B8–B9). For the A stars the mean of V_2, *Lick Observatory Bulletin*, **7**, 20, has been used without correction.

[2] Freundlich, *Physikalische Zeitschrift*, **20**, 561, 1919, has attempted to show that the K-term may be identified with the displacement of spectral lines toward the red required by the theory of relativity. This displacement may be involved, but results given later in this paper indicate that other factors must also play a part. See footnote p. 212.

especially when it is remembered that they form a local aggregation[1] and that their masses are among the largest known.

There are suggestions of systematic divergencies in both sets of residuals, but by forming the means of the two series of values of the energy constant, these disappear, and the average deviation is reduced to 5 per cent.

Were the condition of equipartition exactly satisfied, we should have

$$\overline{v^2}=3\,\overline{\theta^2}\,.$$

For a Gaussian distribution of the radial velocities

$$\overline{\theta^2}=\frac{\pi}{2}\,\bar{\theta}^2$$

where $\bar{\theta}^2$ is the square of the arithmetical mean radial velocity used in Table IX. For unit mass, $\log \overline{v^2}=3.57$; and since $\log \pi/2=0.20$, $\log \overline{\theta^2}=2.96$. The mean-square velocities therefore appear to have a ratio of 4 to 1 instead of 3 to 1. It is known, however, that the radial velocities do not follow a Gaussian distribution closely.

Approaching the matter from another direction, we have for a random distribution in direction, irrespective of the form of the frequency function of the magnitudes of the velocities,

$$\bar{v}=2\,\bar{\theta}.$$

Substituting into the first of equations (12), we find the condition

$$\log \overline{v^2}=\log \theta^2+0.75$$

where θ^2 denotes the square of the arithmetical mean. The difference in the two members of this equation is 0.06 in the logarithm, or 11 per cent, a much better agreement than before. The remaining difference is doubtless to be attributed to the fact that the velocity distribution is not random; stream motion produces a preferential drift parallel to the galactic plane.

[1] This result follows directly from an analysis of star counts. Seares, *Publications of the Astronomical Society of the Pacific*, **30**, 114, 115, 1918. Shapley in recent papers has accumulated much additional evidence: *Mt. Wilson Contr.*, Nos. 157, 161; *Astrophysical Journal*, **49**, 311, 1918, and **50**, 107, 1919; *Mt. Wilson Communications*, Nos. 54, 64; *Proceedings of the National Academy of Sciences*, **4**, 224, 1918; **5**, 434, 1919.

In spite of the known departures from Maxwell's law, there is, with the exception of the B stars, a surprisingly close approach to a state of equipartition. That the deviations are no larger is remarkable,[1] and suggests that for the stars considered (dwarfs) there is little difference between the different spectral types in the average contribution of stream motion to the total energy of translation. Since the unit of mass—that of the sun—is 1.9×10^{33} grams, and since the velocities are expressed in kilometers per second, the mean kinetic energy of translation is of the order of 3.5×10^{46} ergs.

9. DERIVATION OF MASS FROM THE ENERGY-CONSTANT

The constancy of the energy is apparently such as to justify an attempt to evaluate the masses of stars having luminosities and velocities different from those appearing in Table IX. Thus by reading from the curves referred to above the values of v and θ corresponding to different values of M and combining them with the energy-constant, more or less hypothetical values of the corresponding masses can be obtained.

This procedure assumes that equipartition holds for the stars of high luminosity as well as the dwarfs, for which alone a direct test has been possible. The behavior of the B stars suggests caution in an extension of the principle to other giants; but since the translatory energy of the B's is of the same order as that found for the dwarfs, it is perhaps not too much to expect that the use of the energy constant will lead to masses for later-type giants which will also be of the right order of magnitude.

The results obtained in this manner are given in Table X. The values of log $\mathcal{M}$ are the means of the two values found with the aid of v and θ from the two energy constants, log $\mathcal{M}\bar{v}^2$ and log $\mathcal{M}\theta^2$. To indicate the agreement between the masses derived with the aid of the two velocities, the differences log $\mathcal{M}_v-$log $\mathcal{M}_\theta$ are also given in the table. In general these are reasonably small, although for the group of Ma stars the extreme values correspond to a two-to-one ratio in the masses. The data of Table X express, with more

[1] In this connection see Jeans, *Problems of Cosmogony*, p. 238, 1919.

detail and precision, the correlation between mass and luminosity found by van Maanen.[1]

TABLE X

MASS AND LUMINOSITY

M	F5		G5		K3		Ma+	
	$\log \mathcal{M}$	Diff.	$\log \mathcal{M}$	Diff.	$\log \mathcal{M}$	Diff.	$\log \mathcal{M}$	Diff.
−3	......	....	0.74	−0.08	......	....	0.98	−0.36
−2	0.78	−0.02	.62	+ .02	......	....	.64	− .02
−1	.74	− .02	.51	+ .08	0.72	−0.10	.41	+ .16
0	.68	− .02	.38	+ .10	.45	+ .08	.23	+ .24
+1	.62	+ .02	.28	+ .14	.28	+ .14	0.08	+ .30
2	.51	.00	.17	+ .11	.17	+ .12	......	....
3	.30	− .02	0.06	+ .06	.07	+ .08	......	....
4	0.01	−0.04	9.96	− .02	0.01	+ .04	......	....
5	9.77	....	9.88	− .11	9.94	− .02	......	....
6	......	....	9.81	−0.21	9.89	− .04	......	....
7	......	....	......	....	9.84	− .06	......	....
8	......	....	......	....	9.81	−0.08	......	....
10	......	....	......	....	......	....	9.68	+0.14

The general relation of these results to spectral type and absolute magnitude is best seen by means of the diagram in Figure 2. This was prepared by plotting the data in Table X and reading from the resulting curves the values of M coresponding to integral values of the mass $\mathcal{M}$. These values of M were then plotted in the diagram of Figure 2, along with similar points on the dwarf branch, whose co-ordinates were derived from the second and last columns of Table IV. Through these points were drawn the full-line curves of the figure, which thus represent lines of equal mass.

The curves are highly irregular, the maximum near Go followed by the minimum at Ko–K5 being particularly noteworthy. Much of this disappears, however, when the ordinary visual absolute magnitude is replaced by the bolometric magnitude, which measures the total radiation and is therefore the quantity of physical significance. The reductions to bolometric magnitude (see following section) of interest here are

Go, −0.09; G5, −0.30; Ko, −0.73; K5, −1.30; Ma, −1.60 mag.

[1] *Publications of the Astronomical Society of the Pacific*, **31**, 231, 1919.

The application of these corrections would practically remove the minima from the curves for all but the largest masses, and these

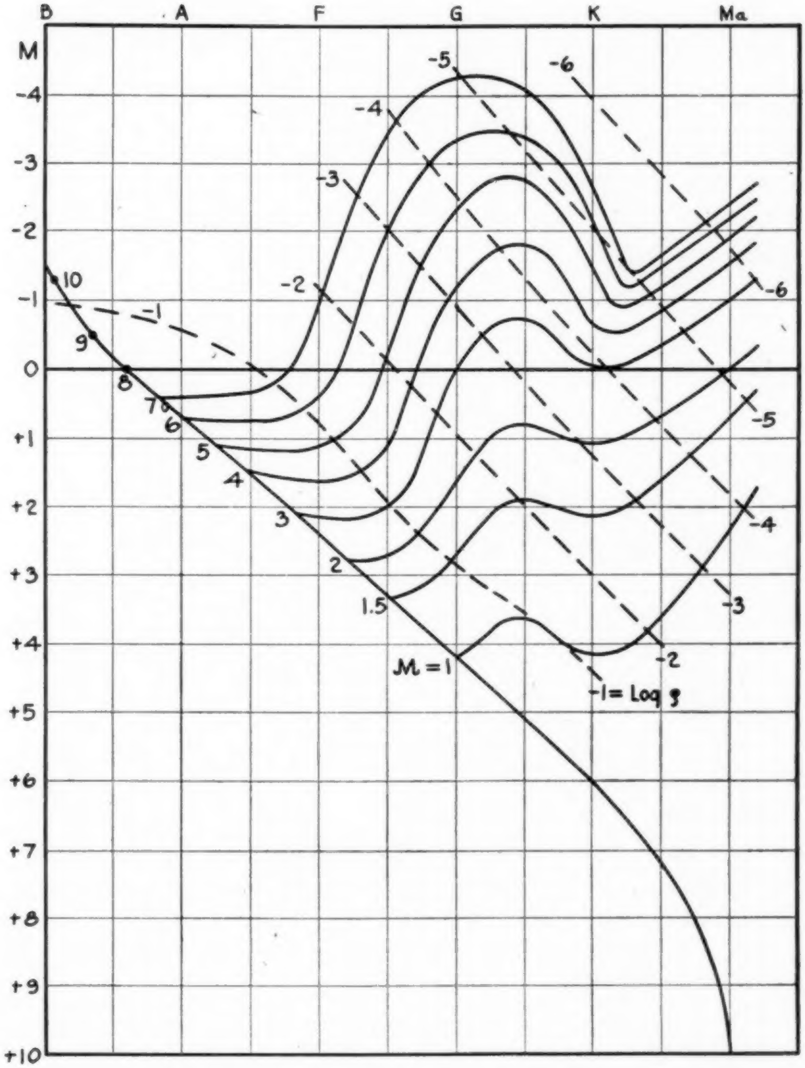


FIG. 2.—Distribution of mass (full-line) and mean density (broken curves) derived from the principle of equipartition. The full-line curve running downward to the right is the line of the maximum frequency of the dwarfs.

are relatively uncertain because they depend more or less upon extrapolations of the data in Table X.

To explain the derivation of the lines of equal density, which are also shown in Figure 2, it is necessary to consider the question of surface brightness.

10. SURFACE BRIGHTNESS

To derive values of surface brightness we may start from Stefan's law

$$E=\sigma T^4 \tag{13}$$

where E represents the total energy radiated per unit area of the surface of a star whose effective temperature is T. Let J be the integrated visual intensity, referred to the same unit of surface, and write $J=\alpha E$, whence

$$2.5 \log E=2.5 \log J-2.5 \log \alpha$$

and, by substitution into (13),

$$2.5 \log J-2.5 \log \alpha=2.5 \log \sigma+10 \log T.$$

Further, let M_B and j represent the total energy and surface brightness, respectively, expressed in magnitudes. The former is the bolometric absolute magnitude, while $M=-2.5 \log (J\times \text{Area})$ is the ordinary visual absolute magnitude. Since α is also the ratio of the visual energy to total energy received from the entire surface of the star,

$$-2.5 \log \alpha=M-M_B$$

and

$$j=M-M_B-10 \log T+\text{const.} \tag{14}$$

where the constant involves $\log \sigma$ and an additional constant depending on the zero point chosen for M_B. Eddington[1] has calculated values of the difference between the visual and bolometric absolute magnitudes $M-M_B$ for a range of temperatures including the normal classes of spectra. Through these values, which are based upon Nutting's measures of visual sensibility for the normal eye, j, by (14), becomes a simple function of T.

[1] *Monthly Notices*, **77**, 605, 1917. Nutting's measures of visual sensibility, which are the basis of Eddington's calculation, are practically identical with the A.I.E. visibility-curve.

Results found by this equation may be checked by an equivalent formula derived by Hertzsprung in 1906. Starting from Planck's law and the visibility measurements of Langley, Abney, and others, he found[1]

$$j_H = +2.3\left(\frac{c}{T}\right)^{0.93} + \text{const.} \tag{15}$$

in which c is the second constant in Planck's formula, for which the value adopted is 14300. Both (14) and (15) involve quadratures, and, starting as they do from entirely different observational data, independently discussed, provide an excellent numerical control. How close the agreement is may be seen from Table XI.

TABLE XI

SURFACE BRIGHTNESS AND TEMPERATURE

T	$M-M_B$	j	j_H	$j-j_H$
2540°	+2.59	+6.32	+6.32	0.00
3000	+1.71	+4.72	+4.68	+ 4
3600	+0.95	+3.17	+3.16	+ 1
4500	+0.35	+1.60	+1.58	+ 2
6000	0.00	0.00	0.00	0
7500	+0.02	−0.95	−0.93	− 2
9000	+0.12	−1.64	−1.62	− 2
10500	+0.31	−2.12	−2.09	− 3
12000	+0.53	−2.48	−2.45	−0.03

The values of $M-M_B$ have been taken directly from Eddington's table. The maximum difference between the results from the two formulae is only 0.04 mag., which is ample.

To connect these results with spectral type, Wilsing's spectral-photometric measures of the reciprocal temperature, c/T, of 199 stars[2] were plotted against their respective types. The ordinates (c/T) of the curve thus obtained are in the second column of Table XII. The corresponding effective temperature, T, and the surface brightness, j, calculated by the foregoing formulae and expressed in magnitudes ($\odot = 0.00$), are in the third and fourth columns, respectively.

[1] *Zeitschrift für wissenschaftliche Photographie*, **4**, 43, 1906.

[2] *Potsdam Publikationen*, No. 74, 1919.

With few exceptions, the stars in Wilsing's list are all giants whose mean absolute magnitudes, with sufficient approximation, may be assumed to be zero. To extend the results to other luminosities, use was made of the Mount Wilson color indices,[1] whose values are in the fifth and last columns of Table XII, those for the

TABLE XII

SPECTRUM, EFFECTIVE TEMPERATURE, AND SURFACE BRIGHTNESS

Sp.	Giants, $M=0$				Dwarfs				
	c/T	T	j	MW CI	c/T	T	j	M	MW CI
B0	1.36	10500°	−2.28	−0.32	......	......	......	......	......
B5	1.43	10000	2.14	−0.17	......	......	......	......	......
A0	1.55	9230	1.89	0.00	......	......	......	......	......
A5	1.76	8110	1.45	+0.19	......	......	......	......	......
F0	2.04	7000	0.88	0.38	......	......	......	......	......
F5	2.35	6080	−0.26	0.62	2.35	6080	−0.26	+3.3	+0.62
G0	2.70	5300	+0.44	0.86	2.48	5770	0.00	4.4	0.72
G5	3.10	4610	1.24	1.15	2.60	5500	+0.25	5.2	0.83
K0	3.70	3860	2.41	1.48	2.93	4880	0.90	5.9	0.99
K5	4.37	3270	3.71	1.84	3.47	4120	1.96	7.1	1.26
Ma	4.65	3080	4.25	1.88	4.30	3330	+3.58	+9.8	+1.76
Mb	4.82	2960	4.58	+1.88	......	......	......	......	......
Mc	4.94	2890	+4.81	......	......	......	......	......	......

dwarfs corresponding to the mean absolute magnitudes in the column immediately preceding. The color indices were reduced to Schwarzschild's absolute system,[2] whose relation to effective temperature is known. This gave the relation between the Mount Wilson values and the temperature for both giants and dwarfs. Owing to various necessary assumptions the temperatures thus found for the giant stars differ slightly from the adopted values given in the first half of Table XII, and this systematic difference was made the basis for reducing the calculated temperatures for the dwarfs to the adopted system of Wilsing. The results, in

[1] These results, which have not hitherto been published, are still to be regarded as provisional. They take account of the important difference in color between giants and dwarfs, which cannot be disregarded in any discussion of the surface brightness of late-type stars. See *Mt. Wilson Comm.*, No. 59; *Proc. Nat. Acad. of Sciences*, **5**, 232, 1919.

[2] *Göttingen Aktinometrie*, B, 29, 1912.

the sixth column of Table XII, were then used to compute the corresponding values of j as before.[1]

The difference in surface brightness between giants and dwarfs is large, and cannot be neglected. Unfortunately, the change in j with absolute magnitude is not well determined, for this must be inferred from the behavior of the color indices, and the data available at present do not extend to stars of the highest luminosity. This means that extrapolation will be necessary, and, in view of the consequent uncertainty, a simple linear relation of surface brightness to absolute magnitude has been used in the calculation of j for all values of M not appearing directly in Table XII.

The reliability of the foregoing results depends upon the possibility of representing the energy distribution in the spectra of stars of different types by means of the black-body radiation formula. It should be noted that it is not at all a question of agreement of the effective temperature, calculated from the radiation formula, with the true mean temperature of the radiating layers in the star's atmosphere. In general these will not agree. The stars are not perfect radiators; further, general and selective absorption deforms the energy-curve to such an extent that effective temperature cannot be an accurate measure of true temperature; but in the visible region the resultant curve seems to agree[2] approximately with a black-body curve, although not with the curve corresponding to the true mean temperature. That being the case, the surface brightness is immediately calculable by the foregoing formulae, provided the effective temperature corresponding to the deformed curve is employed for the computation; and it is of course just this temperature which is given by Wilsing's observations.

[1] The values of j in Table XII may be compared with Öpik, *Astrophysical Journal*, **44**, 296, 1916; Russell, *Publications of the Astronomical Society of the Pacific*, **32**, 307, 1920. Related questions are discussed by Wilsing, *Potsdam Publikationen*, No. 76, 1920; *Astronomische Nachrichten*, **214**, 185, 1921.

[2] See, for example, Wilsing, *Potsdam Publikationen*, No. 74, pp. 17, 18. The mean systematic deviations of the observed values of the energy at different points in the visual spectrum from Planck's formula are small. Further, Milne, *Monthly Notices*, **81**, 378, 1921, finds that the theoretical spectrum for non-selective absorption is practically identical with that corresponding to black-body radiation, the principal difference being a bodily displacement of the energy-curve, corresponding to a reduction of 3 per cent in the constant of Wien's displacement law.

The possibility of systematic error in the values of the temperature is more serious. Those for the B stars given in Table XII are almost certainly too small. The early-type stars in Wilsing's list are not numerous, and the dispersion in the values of c/T is large. In fact, the data would be almost equally well represented by assigning a temperature of 12,000° to B0, which would change j from -2.28 to -2.67. The value for B5 would also be slightly changed, but that for A0 can scarcely be modified without violence to the data.[1]

It will be noted that T for a G0 giant is nearly 500° lower than that for a dwarf of the same spectrum. Some question as to the reliability of Wilsing's measures has been felt because of the low value found for G0 stars as compared with the probable temperature of the sun. But the objects observed by him are giants; with proper allowance for the sun's absolute magnitude (4.85), the corresponding temperature from Wilsing's data would be about 5820°, which is not far from the values usually assigned, and indeed, agrees well with that derived by Abbot[2] from the solar constant (5860°).

A partial control of the values of j in Table XII is afforded by Pease's recent measures of the angular diameters of stars with the interferometer. In this connection, and for later use, it is convenient to insert at this point the following formulae, all easily derived, expressing relations between the various physical constants of a star: angular diameter (d), linear diameter (D), mass ($\mathcal{M}$), density (ρ), apparent (m) and absolute (M) magnitude, and surface brightness (j).

$$\log d = 0.2\,(j-m) - 2.061 \tag{16}$$

$$\log D = 0.2\,(j-m) - \log \pi - 0.030 \tag{17}$$

$$\log D = 0.2\,(j-M) + 0.970 \tag{18}$$

$$D = 107.4\, d/\pi \tag{19}$$

$$j = 5 \log d + m + 10.30 \tag{20}$$

$$\log \rho = \log \mathcal{M} - 3 \log R + 0.14 \tag{21}$$

$$\log \rho = \log \mathcal{M} + 0.6\,(M-j) - 2.77 \tag{22}$$

[1] See also Rosenberg, *Abhandlungen der Kaiserl. Leop.-Karol. deutschen Akademie der Naturforscher*, **150**, No. 2, 1914.

[2] *Op. cit.*, p. 114.

These formulae give d in seconds of arc, D and $\mathcal{M}$ in terms of the solar diameter and mass, and j in magnitudes, the value for the sun being 0; the unit for ρ is the density of water. The numerical constants depend upon the following data for the sun: $D = 865{,}000$ miles, $d = 1920''$, distance from earth 92,930,000 miles, $m = -26.72$, $M = +4.85$. Equations (16) and (17) are the equivalent of formulae already given by Russell.[1]

Proceeding now to a comparison with the results of Pease's measures, we have the data in Table XIII. The "observed"

TABLE XIII

COMPARISON WITH MEASURED ANGULAR DIAMETERS*

	α Orionis	α Boötis	α Scorpii
Sp.	Ma	Ko	Map
m	+0.9	+0.2	+1.2
π	0.″020	0.″095	0.″0085
M	−2.6	+0.1	−4.2
d (Pease)	0.″047	0.″022	0.″040
Obs. j, equation (20)	+4.6	+2.2	+4.5
Cal. j, Table XII	+4.4	+2.4	+4.5
O–C	+0.2	−0.2	0.0
Obs. D, equation (19)	252	25	506
Cal. D, equation (18)	235	27	513
Approx. $\mathcal{M}$ (Fig. 3)	8	3	15
Approx. log ρ (Fig. 3)	−6.2	−3.8	−6.6

* Michelson and Pease, *Mt. Wilson Contr.*, No. 203; *Astrophysical Journal*, **53**, 249, 1921. Pease, *Publications Astronomical Society of the Pacific*, **33**, 171, 204, 1921. The parallax of α Scorpii assumes it to be a member of the Scorpius group. The other two parallaxes are weighted mean values.

values of j derived from the angular diameter by (20) are in the sixth line of the table. Immediately below are the calculated results interpolated from Table XII. It seems unlikely that the mean systematic difference for O−C will exceed one- or two-tenths of a magnitude. The effect of darkening at the limb has not been taken into account, either in the measures of d or in the calculation of j. Nor has the question of a possible differential effect from this source upon the observed and calculated values of j been considered, for it is evident that this cannot be large.

It is of interest, further, to compare the values for the linear diameters calculated from the measured angular diameter by (19) with those derived from (17) or (18) with the aid of the adopted

[1] *Publications of the Astronomical Society of the Pacific*, **32**, 307, 1920.

values of j. The results are in the third and fourth lines from the bottom of table. On the whole, Pease's measures afford an excellent confirmation of the surface brightness found for late-type stars of high luminosity, and at present there seems to be no observational evidence which would justify a modification of the values adopted.[1]

11. DISTRIBUTION OF DENSITY

The densities are now easily calculated by equation (22) with the aid of values of j from Table XII. The results for stars on the dwarf branch, including the B's, are in the fourth column of Table XIV, which also gives values for the mean linear diameters D found by (18).

TABLE XIV

LUMINOSITY, MASS, MEAN DENSITY, AND DIAMETER

(Units for $\mathcal{M}$ and D are values for sun; for ρ, that of water)

Sp.	Dwarf Branch				Giants, $M=0$			Giants, $\mathcal{M}=10$		
	M	$\mathcal{M}$	ρ	D	$\mathcal{M}$	ρ	D	M	ρ	D
B0	−1.60	10	0.04^5	6.8	.....	....	3.2	−1.2	0.08	6
B5	−0.20	8.3	0.20	3.8	8.2	0.26	3.5	−1.8	0.03	8
A0	+0.70	6.0	0.36	2.8	7.0	0.16	3.9	−2.4	0.008	12
A5	+1.50	4.0	0.40	2.4	5.6	0.071	4.8	−3.1	0.002	20
F0	+2.40	2.5	0.40	2.0	4.3	0.025	6.2	−3.6	0.0004	32
F5	+3.35	1.5	0.39	1.8	3.2	0.0078	8.3	−4.0	0.0001	52
G0	+4.35	1.0	0.68	1.26	2.6	0.0025	11.5	−3.9	0.00002	83
G5	+5.20	0.76	1.2	0.96	2.8	0.00087	17	−3.3	0.00001	102
K0	+6.00	0.68	1.3	0.89	3.0	0.00018	28	−2.3	0.00001	107
K5	+7.20	0.62	1.4	0.83	2.6	0.000026	51	−2.2	0.000002	182
Ma	+9.80	0.59	5.4	0.54	2.0	0.0000096	66	−3.0	0.0000006	288

For the determination of the equal-density lines (dotted) shown in Figure 2, values of M were read from each mass line in the diagram for every half-interval of spectrum. The use of (22) then gave the values of log ρ corresponding to each of these points in the sequence of spectral type. Plots of these results—log ρ against spectrum—then led to a series of curves, one for each integral value of the mass (also one for $\mathcal{M}=1.5$), from which were read spectral types corresponding to integral values of log ρ. The

[1] Mr. Pease has kindly placed at my disposal provisional results for three other stars. The observed and theoretical values of j and d, or D, are here also in excellent agreement, although the accidental deviations are larger than those shown in Table XIII.

corresponding points were marked on the diagram and connected by the dotted lines shown in the figure.

Within the limits considered, and with the exception of that for $\log \rho = -1$, the curves of equal density are sensibly linear; the regularity of their spacing is remarkable. Including the line for $\log \rho = -1$ from A5 onward through the later types, the mean density corresponding to any given spectrum and absolute magnitude is represented by the formula

$$\left.\begin{array}{c}\log \rho = -1.22\,S + 0.57\,M + 1.11 \\ (S = 0 \text{ for B0, 1 for A0, etc.})\end{array}\right\} \qquad (23)$$

with a probable error of about 15 per cent, over a range in ρ extending from 0.1 to 0.000001.

The question now arises as to what means can be found for testing the validity of these results. Some control is obviously necessary, for with the exception of the masses and densities along the dwarf branch, the whole scheme of mass and density distribution depends upon the assumed applicability of the principle of equipartition. In this connection, the eclipsing variables, many of which are known to be of exceptionally low density, immediately suggest themselves as suitable for a test. The weak point, however, is that we do not know their absolute magnitudes, except in a very few cases. Any attempt to use hypothetical absolute magnitudes fails, because it involves an assumption for the mass, and this, as shown by (22), is equivalent to a direct assumption for the density.

Some general indications, however, can be obtained. With slight revisions of spectral classification, Shapley's data[1] for the densities of eclipsing variables give the following mean values:

Sp.	ρ	No. Stars
B6	0.12	15
A0+	0.20	48
F(dwarfs)	0.35	9

For these groups of B and A stars the values of $\log \rho$ are -0.9 and -0.7, respectively. From Figure 2 the corresponding values of M are seen to be -0.6 and $+0.2$. These are of the order to be

[1] *Contributions from Princeton University Observatory*, No. 3, p. 124, 1915.

expected, and are obviously possible values. The value $\rho = 0.35$ for the F-type dwarfs agrees closely with that found here for F5, namely, 0.39. For the dwarfs the dispersion in absolute magnitude is so small that lack of knowledge of the luminosities gives rise to no appreciable error.

These results are all satisfactory, as far as they go, but they do not reach that part of the diagram which is involved in the greatest doubt, namely the region of very high luminosity and exceptionally low density. On the other hand, if the validity of the mass and density distribution can once be established, the densities of the eclipsing variables should give a reliable determination of their distances. Fortunately, the uncertainty can be removed to a large extent with the aid of the Cepheid variables.

12. REVISION OF MASSES AND DENSITIES BY MEANS OF CEPHEID VARIABLES

There are still difficulties connected with the pulsation hypothesis as a means of explaining Cepheid variation, although there seems little doubt that, in some form or other, it must be used to account for the changes in brightness of these stars. That being the case, the mean density must vary inversely as the square of the period. Hence[1]

$$\log \rho = -2 \log P + \text{const.} \tag{24}$$

Further, the period-luminosity relation[2]

$$M = f(P) \tag{25}$$

[1] The exact form of the relation of density to period as applied to Cepheids is possibly open to some question. Emden, *Gaskugeln*, p. 451, 1907, finds for ellipsoidal oscillations of a polytropic gas sphere

$$P \propto (m\,\rho)^{-\frac{1}{2}}$$

where m is the order of the harmonic oscillation; the important vibration corresponds to $m=2$. Eddington, *Monthly Notices*, **79**, 2, 1918; 177, 1919, discusses pulsations which are symmetrical about the star's center and finds a result equivalent to (24), with the exception that the constant is replaced by a slowly varying function of the specific heats and the ratio of radiation-pressure to gravitation, the latter increasing with the mass. For a first approximation we may adopt (24).

[2] Shapley, *Mt. Wilson Contr.*, No. 151; *Astrophysical Journal*, **48**, 89, 1918.

determines the absolute magnitude. With M and the spectral type as arguments, log ρ and $\mathcal{M}$ can be interpolated from the mass-density diagram. Then,

a) The interpolated values of log ρ may be compared with those calculated by (24), the constant being adjusted to give the best representation of the densities of the entire group of stars. Since the range in ρ as determined by (24) is from about 0.1 to 0.00001, the control on the densities should be effective.

b) Having determined the constant in (24), we may write (22) in the form

$$\log \mathcal{M} = -2 \log P - 0.6 \, (M - j) + \text{const.} \tag{26}$$

which expresses the mass of a Cepheid variable as a function of its period and surface brightness, for, by (25), M is a function of the period alone. The values of $\mathcal{M}$ interpolated from the diagram may be compared with those calculated by (26). This checks the distribution of mass.

c) Finally, with spectral type and log ρ calculated from (24), we may interpolate M from the diagram. The absolute magnitudes thus found may then be compared with the period-luminosity relation (25). This is not an independent test, but exhibits the results of (*a*) in a different form.

Because of the enormous range in the density, the application of these tests will mean little except in the case of stars of accurately determined spectra. This makes trouble at once, for the type of a Cepheid, in so far as determined from the hydrogen lines at least, varies with its brightness, as was first clearly shown by Pease's[1] observations of RS Boötis. Isolated determinations of spectrum based on the appearance of these lines are therefore unsuitable for the comparisons. Adams and Joy,[2] on the other hand, have shown that many spectral characteristics change little or not at all as the light of the star varies; their "estimated" spectral classes are practically independent of phase.

[1] *Publications of the Astronomical Society of the Pacific*, **26**, 256, 1914.

[2] *Mt. Wilson Comm.*, No. 53; *Proceedings of the National Academy of Sciences*, **4**, 129, 1918.

The list of spectroscopic parallaxes[1] includes twenty-nine Cepheids for which estimated types are available. One of these, RY Boötis, although usually classed as a Cepheid, does not show Cepheid spectral characteristics and has an absolute magnitude at variance with that derived from the period-luminosity relation. Further, three cluster-type Cepheids, SU Aurigae (8.7), SU Draconis (9.2), and SW Draconis (10.0), and the long-period Cepheid V Vulpeculae (8.6) are so faint that the types assigned are less certain than the others. The classification of faint stars, observed with low dispersion, is likely to be influenced to some extent by the hydrogen lines. The case of V Vulpeculae, moreover, is peculiar. The light-curve is irregular; and the period-luminosity relation leads to a value of M which not only differs widely from that derived from the spectrum, but falls outside the limits of the mass-density diagram. These five stars were therefore excluded,[2] leaving a total of twenty-four.

These data are supplemented by Shapley's[3] results for twenty Cepheids, based on systematic observations of spectral characteristics that do vary with the changes in light. The median value of the type, which was used for the discussion, is about 0.3 of a spectral interval earlier than the estimated spectrum.

This difference will lead to different values of the constants in the foregoing formulae, unless the value of M and the spectral type are referred to the same phase. Thus, the estimated type agrees with the hydrogen-line type at, or very near, the time of minimum brightness (estimated *minus* hydrogen-line type at minimum $= -0.1$ spectral interval). Hence for the comparisons, the minimum, and not the median M (the period-luminosity relation gives the latter), should be associated with the estimated spectra. The simplest procedure, however, is to use the median M and subtract 0.3 of an interval from the estimated spectra before

[1] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[2] It was not noted until after the comparisons had been finished that, on the criterion of brightness, W Serpentis (9.0) should also have been excluded.

[3] *Mt. Wilson Contr.*, No. 124; *Astrophysical Journal*, **44**, 273, 1916. In general the means of the limiting values of the type as given by Shapley have been adopted. Deviations from this rule occur in a few cases where the distribution of the observations is such that the mean does not correspond to median brightness of the variable.

making interpolations from the diagram. Sixteen stars are common to the two lists, but owing to the different methods of classification used, the duplicate observations have been treated as separate stars.

The results of the application of tests (*a*), (*b*), and (*c*) are summarized in Table XV, which gives the range in log ρ, log $\mathcal{M}$,

TABLE XV

REPRESENTATION OF MASSES AND DENSITIES
EQUATIONS (24) TO (26) AND FIGURE 2

Test	Range	Average Difference	
		Est. Sp.	H.L. Sp.
(*a*) log ρ	-1.1 to -4.8	± 0.15	± 0.18
(*b*) log $\mathcal{M}$	0.46 to 1.28	0.15	0.16
(*c*) M	-0.3 to -4.7	± 0.30	± 0.32

and M over which the comparisons were made, and the average differences resulting therefrom. Thus, on the basis of estimated spectra, formula (24) represents the density distribution in Figure 2 over a range in log ρ from -1.1 to -4.8, with an average difference of ± 0.15. Similarly, the values of M interpolated from the diagram in the interval -0.3 to -4.7 agree with those from the period-luminosity relation within ± 0.30 mag. We conclude at once that the masses and densities derived with the aid of the principle of equipartition are of the right order of magnitude, at least as far as type G3, which is the latest occurring among the Cepheids used.

A classification of the differences in log ρ and log $\mathcal{M}$ according to M, which is also more or less a classification according to spectrum, revealed, however, the presence of systematic divergences. These run from about -0.18 (at $M = -0.7$, A8) to $+0.18$ (at $M = -3.8$, F8) in the logarithms of both quantities. The algebraic signs mean that the range of mass in Figure 2, for a given interval in M, is too small, that the masses of the most luminous Cepheids are larger, and those of less luminous stars smaller, than the values given by the diagram. The density lines, on the other hand, are too closely spaced.

These differences were made the basis of a readjustment of the lines of mass and density. The percentage divergence in ρ and $\mathcal{M}$

is the same, as it must be by (22), but since the scale for $\mathfrak{M}$ is much larger than that for ρ (in passing from one curve to the next the mass changes by one unit while ρ changes tenfold), the adjustments were made in the mass-curves, the densities being computed by (22) as before. The masses and densities along the dwarf branch, of course, remain unchanged.

The result of several trials is the diagram shown in Figure 3. A repetition of tests (*a*), (*b*), and (*c*) gave the differences shown in the last six columns of Table XVI, the first column under each of the headings log ρ, log $\mathfrak{M}$, and M referring to estimated spectra, the second to those based on the hydrogen lines. The average differences have been reduced to about two-thirds their original values, while the run in the differences has disappeared. Thus for four groups of eleven stars each, the mean systematic values are:

M	−1.0	−2.1	−2.7	−3.6
Sp.	F0	F6	F7	F8
log ρ	−0.01	−0.03	+0.02	+0.02
log $\mathfrak{M}$	−0.02	−0.01	−0.04	0.00

The comparison of the interpolated values of M with the period-luminosity relation is shown graphically in Figure 4.

In this connection it will be noted that values of M interpolated from the diagram with log ρ and the type as arguments are sensitive to uncertainties in the spectra. The relations are such that the average difference ±0.2 in M (Table XVI) corresponds to 0.1 of a spectral interval (see equation [23]). The normal uncertainty of classification therefore accounts for nearly all the deviation shown by individual stars in Figure 4.

The principal change in the diagram has been the depression of the mass-curves near G0 and the change in their curvature in the region of the early types. Since the data for the Cepheids do not extend beyond G3, the lines through types G to M have been left much as in the original diagram. As will be seen later, there is some evidence favoring the reality of the minimum near K0. If the reductions to bolometric magnitude given on page 194 be applied, the variation of mass with spectrum becomes fairly

TABLE XVI

MASSES AND DENSITIES OF CEPHEID VARIABLES

Star	Spectrum		P	Calculated				Calculated *minus* Fig. 3					
	Est.	H.L.		M	$\log \rho$	$\mathfrak{M}$		$\log \rho$		$\log \mathfrak{M}$		M	
α Ursae Minoris	F9	....	3.97	−1.8	−3.10	5.0		+0.01		+0.03		0.0	
TU Cassiopeiae	....	F4.5	2.14	−1.2	−2.56		5.5		+0.08		+0.11		−0.1
SU Cassiopeiae	F4	F1.5	1.95	−1.2	−2.48	3.0	2.9	−0.17	−0.09	−0.22	−0.24	+0.4	+0.3
SZ Tauri	F8p	F6	3.15	−1.6	−2.90	4.6	4.4	+0.01	+0.09	0.00	−0.01	0.0	−0.3
RX Aurigae	F9p	....	11.63	−3.4	−4.04	5.2		−0.18		−0.21		+0.3	
T Monocerotis	G1p	F8.5	27.01	−4.7	−4.76	13.8	13.2	0.00	+0.03	+0.06	+0.04	0.0	−0.1
RT Aurigae	F8p	F5	3.73	−1.8	−3.04	5.0	4.8	−0.03	−0.05	+0.02	−0.01	0.0	+0.1
W Geminorum	G0p	F6.5	7.92	−2.8	−3.70	5.8	5.5	+0.01	−0.06	−0.07	−0.09	0.0	+0.1
RS Boötis	....	A4	0.38	−0.3	−1.06		9.6		+0.18		+0.17		−0.2
ζ Geminorum	G0p	....	10.15	−3.2	−3.92	6.0		−0.06		−0.12		0.0	
X Sagittarii	F9p	F5	7.01	−2.6	−3.60	4.8	4.6	−0.14	−0.21	−0.12	−0.15	+0.3	+0.5
Y Ophiuchi	....	F9	17.11	−4.0	−4.36		14.4		+0.13		+0.16		−0.2
W Sagittarii	G0p	F6	7.60	−2.7	−3.66	5.5	5.2	0.00	−0.12	−0.07	−0.10	0.0	+0.3
Y Sagittarii	G2p	F9	5.77	−2.3	−3.42	9.6	9.1	+0.24	+0.27	+0.24	+0.22	−0.5	−0.5
U Sagittarii	G0p	....	6.74	−2.5	−3.56	6.0		0.00		0.00		0.0	
YZ Sagittarii	G1p	....	9.55	−3.1	−3.86	9.1		+0.05		+0.08		−0.2	
TT Aquilae	G2p	....	13.75	−3.6	−4.18	11.5		+0.13		+0.11		−0.3	
RR Lyrae	F0p	A5.5	0.57	−0.4	−1.42	6.6	6.3	+0.09	+0.02	+0.09	+0.03	−0.2	−0.1
U Aquilae	G0p	F9	7.02	−2.6	−3.60	5.5	5.2	+0.01	+0.19	−0.05	−0.07	0.0	−0.4
XZ Cygni	....	A3	0.47	−0.3	−1.24		5.5		−0.10		−0.08		+0.2
U Vulpeculae	G0p	F7	7.99	−2.8	−3.70	5.8	5.5	+0.01	−0.01	−0.07	−0.09	0.0	0.0
SU Cygni	F7p	F4	3.85	−1.8	−3.08	3.5	3.3	−0.17	−0.19	−0.16	−0.18	+0.3	+0.4
η Aquilae	F9p	F7	7.18	−2.6	−3.62	4.6	4.4	−0.11	−0.03	−0.14	−0.15	+0.3	+0.1
S Sagittae	G1p	F8.5	8.38	−2.9	−3.74	7.9	7.6	+0.12	+0.15	+0.06	+0.04	−0.2	−0.3
X Cygni	G3p	....	16.38	−3.9	−4.32	19.0		+0.24		+0.28		−0.5	
T Vulpeculae	F9	F5	4.44	−2.0	−3.20	5.2	5.0	+0.01	−0.11	+0.02	−0.02	0.0	+0.2
δ Cephei	G0	F6	5.37	−2.2	−3.36	6.3	6.0	+0.05	−0.07	+0.08	+0.05	−0.1	+0.1
W Serpentis	F9p	....	14.15	−3.7	−4.20	6.3		−0.19		−0.16		+0.4	
Average difference	....	....	....	....	....	....		±0.08^5	±0.11	±0.10	±0.10	±0.17	±0.22

regular, the course of the topmost line ($\mathcal{M} = 10$) being indicated by the dotted curve in Figure 3.

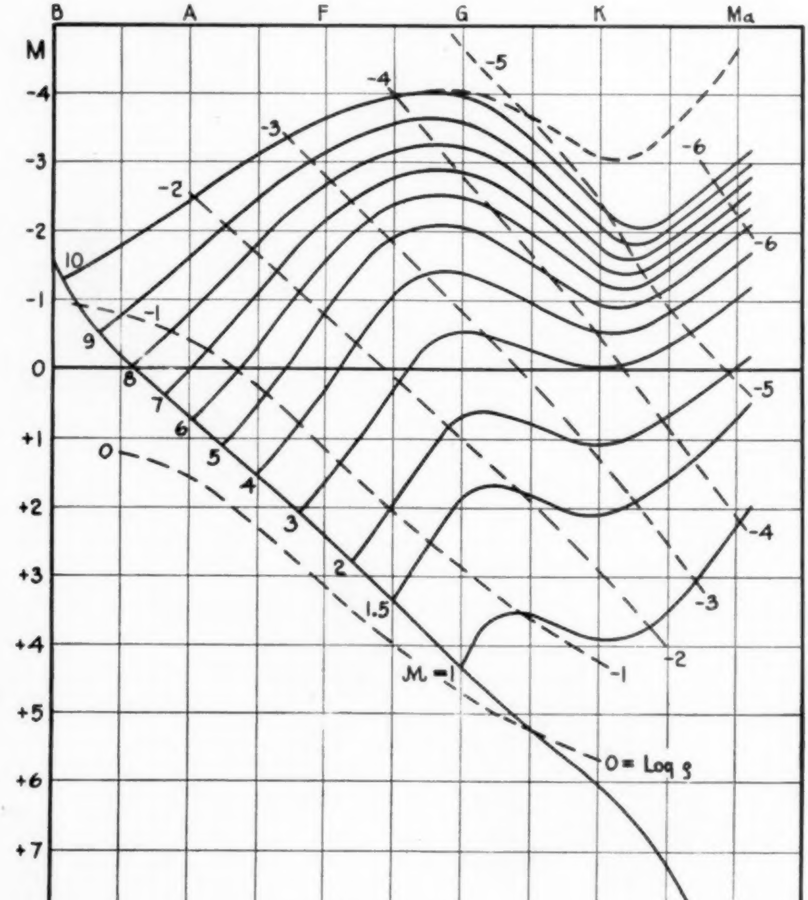


FIG. 3.—Distribution of mass and mean density, revised with the aid of Cepheid variables. The dotted curve at the top indicates the reduction to bolometric absolute magnitude for late-type stars of mass 10.

The curves of equal density are little changed. They still remain sensibly linear, but are no longer so closely parallel. Formula (23) has not wholly lost its applicability, for it still expresses approximately the relation of density to spectrum and absolute magnitude, though less accurately than before.

It will be noted that the densities along the dwarf branch are consistent with the values demanded by the density-distribution defined by the lines log $\rho=0, -1, -2$, etc. In other words, if we

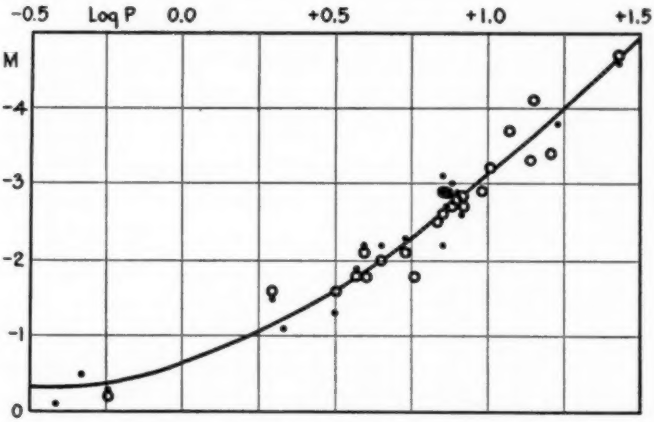


FIG. 4.—Absolute magnitudes of Cepheids interpolated from Fig. 3 with spectral type and log ρ from equation (24) as arguments. Circles, "estimated" spectra by Adams; points, hydrogen-line spectra by Shapley. The curve is Shapley's period-luminosity relation.

interpolate, say, for the value log $\rho=-0.4$ ($\rho=0.4$) along the line $\mathcal{M}=6$, we find the point where it belongs, namely, on the dwarf branch. This detail is of some importance because it shows that the constants in the formulae for log ρ and log $\mathcal{M}$ have been correctly determined. The spacing of the lines is fixed by the periods of the variables; their position as a group, by the condition that they must be consistent with the densities adopted for the dwarfs, whose values are independent of any assumption as to equipartition.

Presumably the diagram in Figure 3 is to be preferred to that in Figure 2. Certainly that is the case when Cepheid variables are concerned; and it seems likely that this holds also for the

stars at large. The mass lines in Figure 2 pass *through* the points defined by the energy-constant and the velocities; no allowance for observational uncertainty by smoothing between the types was attempted; and with the exception of the terminal points on the dwarf branch everything preceding F5 is an extrapolation. On the other hand, an assumption of comparability is here involved, and it is perhaps worth remarking that if we reject the original diagram, we may thereby exclude relations between mass, density, luminosity, temperature, etc., which differentiate stars of constant light from Cepheid variables.

In view of the tentative character of the results, a detailed tabulation of the mass and density relations is unnecessary; Figure 3 will answer all practical purposes to which the results can be adapted. The data for the dwarf branch given in the first part of Table XIV, which are the most reliable, have already been referred to. To these have been added, by way of numerical illustration, results for the giants of zero absolute magnitude and for those whose masses are equal to 10.[1]

As a further illustration, the masses and densities corresponding to the spectra and absolute magnitudes of the individual stars measured by Pease with the interferometer have been given in the last lines of Table XIII. As a source of error, the values of M for these stars are probably as serious in respect to mass and density as the uncertainties of Figure 3 itself. Equation (22) shows that an error of 0.5 in M means an uncertainty factor of 2 in the density, whereas it seems unlikely that values from the diagram should be in error by so large an amount. In the case of an individual star, dispersion may of course enter, since the diagram gives mean

[1] Since the displacement of spectral lines required by relativity is proportional to $\mathcal{M}/R$ (R = radius of star) the results in Table XIV have a direct bearing on this question. The values of $\mathcal{M}/R$ do not run parallel to those of the K-term. Hence this term cannot be wholly identified with the relativity displacement, although the latter may of course be involved. As an illustration, the K-term for the B stars is about four times that for the A's. The ratio of the displacements on the other hand would be about 1.3. The most extreme deviation, however, is found by comparing the B's with the giant M stars. Their K-terms are of the same order of magnitude, whereas the red-displacement of the lines in M stars should be of the order of 0.01 that of the B's. See footnote No. 2, p. 191.

values; but the probability of the occurrence of large deviations from the mean is small.

13. ON DISPERSION IN THE MASSES OF CEPHEIDS

From Shapley's work it appears that, for the variables in globular clusters at least, the deviations in M from the values determined by the period-luminosity relation are of the same order as the observational errors in the apparent magnitudes,[1] which of course must be used for any comparison. Practically, therefore, M is uniquely determined by P. Hence, by (26), the dispersion in the mass is related to that arising from such differences in j as may be associated with the same value of P, and it is an observational fact that the latter are small.

Direct evidence is also afforded by the comparison of the values of $\mathcal{M}$ for the Cepheids calculated by (26) with those interpolated from Figure 3. The average difference in log $\mathcal{M}$ is ± 0.10, corresponding to about 25 per cent in $\mathcal{M}$. But what part of this may be attributed to uncertainty in the spectrum?

With proper allowance for the systematic difference of 0.3 between the hydrogen line and estimated spectra, the internal agreement is such as to indicate an average uncertainty of 0.05 of an interval. Assume an adopted value of the type to be too late by this amount. From (26)

$$\text{Mod.}\ \frac{d\mathcal{M}}{\mathcal{M}} = +0.6\ dj. \tag{27}$$

For the region F0–G0, j increases about 0.07 for the assumed increase in the spectrum. Hence, the calculated $\mathcal{M}$ will be 10 per cent too large. On the other hand, we find from Figure 3, for F5, $M = -2.5$, as an average, that the same error in the spectrum leads to an interpolated $\mathcal{M}$ about 3 per cent too small. The resulting errors are in opposite directions; whence, in the difference, the uncertainty becomes 13 per cent.

This average percentage error, combined with the dispersion and the errors of interpolation, must equal the average deviation

[1] *Mt. Wilson Contr.*, No. 151, p. 26; *Astrophysical Journal*, **48**, 114, 1918.

of 25 per cent from the mean masses of the diagram. Neglecting the interpolation errors, whose inclusion would make the dispersion still smaller, we find the probable dispersion $r(\mathcal{M})=19$ per cent. As far as the evidence goes, we should therefore expect one-half the Cepheids to have masses lying between 0.8 and 1.2 times the mean mass. Since the Cepheids are a very special class of objects, it would be too much to assume that this result holds for all classes of stars (see Section 7).

14. MASSES AND DENSITIES OF CEPHEID VARIABLES

The calculated values of the masses and densities used for the comparisons with the diagram of Figure 3 have been collected in the sixth and seventh columns of Table XVI, while their deviations from the adopted distribution are in the following columns. The formulae used are:

$$\log \rho = -2 \log P - 1.90 \tag{29}$$

$$\log \mathcal{M} = -2 \log P - 0.6\ (M-j) + 0.87 \tag{30}$$

in which the constants are the means for the two sets of data, the individual values differing by only 0.02. In case the spectral type is the median value derived from the hydrogen lines, the median value of M, given directly by the period luminosity relation, is to be used. For estimated spectra, the minimum M is required, although it is simpler to subtract 0.3 from the estimated spectrum and use the median M.

The values of both density and mass depend upon the assumed constancy of the numerical term in (29), but this assumption apparently leads to results of the right order, although for stars of short period the percentage deviation may be considerable.

For individual stars the calculated values are probably to be preferred to those interpolated from Figure 3. This follows not merely because of the dispersion in mass, but also because the differences for stars having determinations based on both estimated and hydrogen-line spectra are almost always of the same sign. This indicates real deviations from the mean distribution, in addi-

tion to those arising from the unavoidable errors of spectral classification. In such cases the values from (29) and (30) are likely to be nearer the truth than those derived from the diagram.

The results in Table XVI, both calculated and interpolated, may be compared with the masses and densities derived by Eddington[1] for certain Cepheids on the basis of his theory of radiative equilibrium. The agreement shown by Table XVII is remarkable

TABLE XVII

COMPARISON WITH EDDINGTON'S RESULTS FOR CEPHEID VARIABLES

Star	Mass			Mean Density		
	Eddington	Eq. (30)	Fig. 3	Eddington	Eq. (29)	Fig. 3
Y Ophiuchi	13.0	14.4	10	0.000080	0.000044	0.000032
ζ Geminorum	7.4	6.0	7.9	.00046	.00012	.00014
S Sagittae	6.1	7.8	6.9	.00019	.00018	.00013
W Sagittarii	5.5	5.4	6.5	.00041	.00022	.00025
η Aquilae	5.2	4.5	6.2	.00033	.00024	.00028
X Sagittarii	5.1	4.7	6.4	.00025	.00025	.00038
Y Sagittarii	4.3	9.4	5.5	.00027	.00038	.00021
δ Cephei	4.0	6.2	5.3	.00052	.00044	.00045
T Vulpeculae	3.6	5.1	5.1	.00072	.00063	.00071
SU Cygni	3.1	3.4	5.0	.00160	.00083	.0013
RT Aurigae	3.0	4.9	4.8	.00110	.00091	.0010
SZ Tauri	2.8	4.5	4.5	.0015	.0013	.0011
SU Cassiopeiae	2.3	3.0	5.0	.0028	.0033	.0045
RR Lyrae	1.7	6.4	5.6	0.015	0.038	0.033

when it is considered that the two methods approach the problem from entirely different directions. There are a few cases of serious discordance, and some systematic divergence, but nothing so large as might have been expected. The agreement with (29) has been commented upon by Eddington himself.

The geometrical mean mass of the twenty-four Cepheids in Table XVI is 6.2, which happens to agree almost exactly with that derived from the principle of equipartition. This can scarcely be more than accidental, however, for the velocities of only fifteen of these stars are available.

[1] *Monthly Notices*, **79**, 5, 1918.

15. AVERAGE HEAT CONTENT AND CEPHEID VARIATION

On the basis of the theory of radiative equilibrium Eddington[1] finds for the central temperature of a star

$$T_c \propto \rho^{\frac{1}{3}} \mathcal{M}^{\frac{2}{3}} \beta m \tag{31}$$

where ρ is the mean density, β the ratio of radiation pressure to gas pressure, and m the atomic weight which is here assumed to be constant. The formula may also be written

$$T_c \propto \frac{\beta \mathcal{M}}{R} \tag{32}$$

R being the radius. $R = D/2$ is given by (18) or an equivalent expression; β is a function of $\mathcal{M}$, and $\mathcal{M}$ in turn appears in his formulae as a function of the luminosity alone, which can be obtained from (25). T_c can therefore be calculated for any star. For thirteen Cepheids with periods ranging from 2 to 17 days Eddington[2] finds that T_c/β is sensibly constant. There are accidental deviations, but no progressive change. Moreover, for two theoretical stars with periods of 4.5 and 30.8 days the values of T_c/β are practically identical. This suggests as a possible condition for Cepheid variation

$$\frac{\mathcal{M}}{R} = \text{const., or } \rho \mathcal{M}^2 = \text{const.} \tag{33}$$

Since $\mathcal{M}^2/R$ is proportional to the total heat-content of a star,[3] the physical significance of (33) is that the average heat-content per unit mass is a constant, or, in another form, that the gravitational potential at the surface of the star is constant for the Cepheids. Shapley[4] has discussed this condition from the standpoint of the relation of period to absolute magnitude and finds agreement with the observed period-luminosity relation (25). We may proceed somewhat differently by combining (24) and (26) in accordance with (33). This gives the theoretical period-luminosity relation

$$M = -5 \log P + j + \text{const.} \tag{34}$$

[1] *Monthly Notices*, **77**, 601, 1917. [2] *Ibid.*, **79**, 5, 1918; 181, 1919.

[3] Jeans, *Problems in Cosmogony*, p. 191, 1919.

[4] *Mt. Wilson Contr.*, No. 190, p. 7; *Astrophysical Journal*, **52**, 79, 1920.

A comparison of the values of M from (34) for the twenty-eight stars in Table XVI with those from the observed relation, (25) above, gives results which are shown graphically in Figure 5. The agreement is not good, the divergence amounting to about a magnitude between $\log P = 0.25$ and 1.5, with even larger differences for the cluster-type Cepheids, which are not shown in the diagram.

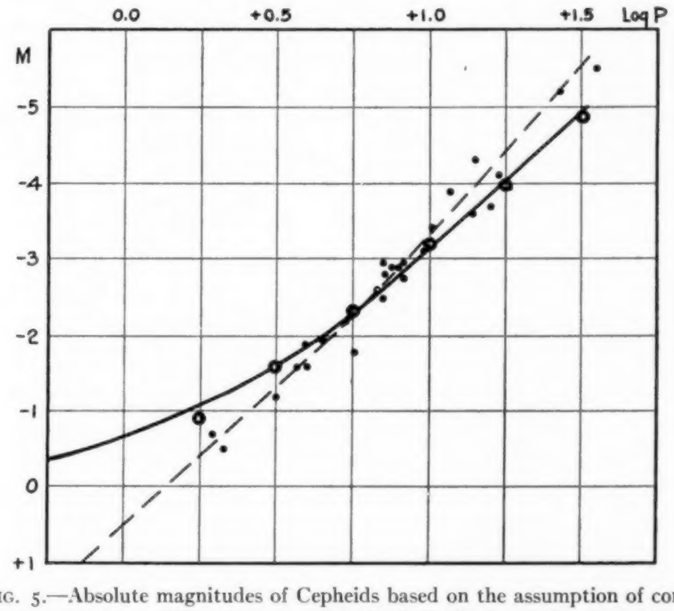


FIG. 5.—Absolute magnitudes of Cepheids based on the assumption of constant average heat content. The points are values for individual stars from equation (34). The circles, from equation (36), show the agreement with Shapley's period-luminosity relation when the term involving the specific heats in Eddington's expression for the density is included.

It will be noted that (33) is an empirical result which follows from an application to Cepheids of the period-luminosity relation (25) and the general theory of radiative equilibrium, but its derivation does not in any way involve the theory of Cepheid variation itself. If valid, it ought, however, in combination with the latter theory, to lead back to relation (25). But the comparison, as made, does not give complete agreement. Since the theory of Cepheids has entered only through (24), the difficulty

presumably lies with that equation, which has been assumed to be applicable. If instead we use the corresponding relation derived by Eddington[1] himself, the difficulty disappears. For (24) we must substitute

$$\log \rho = -2 \log P - 2 \log (\gamma\alpha)^{\frac{1}{2}} + \text{const.} \tag{35}$$

where $(\gamma\alpha)^{\frac{1}{2}}$ is a function of β and of the ratio of the specific heats, and changes slowly with the mass. With this emendation, (34) becomes

$$M = -5 \log P + j - 5 \log (\gamma\alpha)^{\frac{1}{2}} + \text{const.} \tag{36}$$

The factor depending on the specific heats enters to the fifth power and thus becomes important. Its neglect in (34) causes the divergence shown by Figure 5.

The necessary corrections are easily calculated, however, with the aid of Table V of Eddington's paper. The values of $1-\beta$ required for the interpolation of $(\gamma\alpha)^{\frac{1}{2}}$ are given in *M.N.*, **77**, 602, 1917, under the heading "Molecular Weight 2," with the mass as argument. For $\mathcal{M}$ we may use the values in Table XVI, or more consistently with the present procedure, we may calculate $\mathcal{M}$ from

$$\log \mathcal{M} = -0.2\,(M - j) + \text{const.} \tag{37}$$

which follows at once from (18) and the condition $\mathcal{M} \propto R$ given by (33). The latter method was adopted, the constant in (37) being determined so that the mean $\mathcal{M}$ from (37) agrees with that from the diagram in Figure 3. We are interested only in the variation of $(\gamma\alpha)^{\frac{1}{2}}$, which is practically the same for all possible values of the ratio of specific heats. $\Gamma = 1\frac{4}{9}$ was used, however, since Eddington's results show that this agrees well with the data of observation. With the constant in (36) equal to 1.48, the values of $-5 \log (\gamma\alpha)^{\frac{1}{2}}$ are as shown in the third column of Table XVIII. The sums of the remaining terms on the right of (36) are given in the second column under the heading M_o. These are the co-ordinates of the dotted curve in Figure 5 and were read directly from a large-scale diagram. The calculated values of M from (36) (fourth column) when compared with those from (25) give the residuals in the last

[1] *Monthly Notices*, **77**, 15, 1918.

column of the table. For values of P between 3 and 30 days the agreement is very close. For $P < 2$ days there is still a large divergence, but that is to be expected. For all but the cluster-type Cepheids, therefore, the theoretical relation (36) is practically the equivalent of the observed period-luminosity relation (25).

TABLE XVIII

THEORETICAL AND OBSERVED PERIOD-LUMINOSITY RELATION

Log P	M_0	$-5 \log (\gamma\alpha)^{\frac{1}{2}}$	M (36)	M (25)	O−C
0.25	−0.45	−0.45	−0.90	−1.08	−0.18
0.50	−1.30	−0.29	−1.59	−1.58	+0.01
0.75	−2.25	−0.08	−2.33	−2.26	+0.07
1.00	−3.34	+0.15	−3.19	−3.15	+0.04
1.25	−4.43	+0.45	−3.98	−4.05	−0.07
1.50	−5.54	+0.68	−4.86	−4.95	−0.09

Shapley's discussion referred to above starts with the assumption of constant heat-content and leads to a theoretical relation which agrees well with (25), but does not involve Eddington's theory of Cepheids because his comparison is based on equation (24) instead of (35). The close agreement found by Shapley is due partly to the fact that his final formula neglects $2.5 \log (\gamma\alpha)^{\frac{1}{2}}$ instead of $5 \log (\gamma\alpha)^{\frac{1}{2}}$, which by itself would lead to a divergence from (25) equal to one-half that shown in Figure 5, and partly to his readjustment of the mean atomic weight.

The matter is of interest from two or three points of view: First, because it shows the consistency of Eddington's theory of Cepheid variation with the general theory of radiative equilibrium, in that it is necessary to include the term depending on the specific heats in order to work back to the observed period-luminosity relation which was one of the premises; and second, because it exhibits the empirical result (33) in another form and possibly gives weight to the suggestion that constant average heat-content may be a determining circumstance in Cepheid variation.

The fact that the term which has been under discussion enters to so high a power subjects Eddington's theory of Cepheid variation to a severe test. Something of the sort is also true of the

empirical condition (33). Assume for a moment that the relation $\mathcal{M} \propto R$ is not exact, and suppose that all the variations which occur are thrown into the mass. The maximum systematic deviation in log $\mathcal{M}$, by (18), (22), and (37), will then be equal to 0.4 of that shown by the differences in the last column of Table XVIII, which is well within the uncertainty of the comparison.

This close agreement justifies the mention of other relations which follow at once from the assumption of constant average heat-content, combined with the following equations of radiative equilibrium:[1]

$$\left.\begin{aligned} T_e^4 &\propto g(1-\beta) \propto \mathcal{M}^{\frac{1}{3}}\rho^{\frac{2}{3}}(1-\beta) \\ T_c &\propto \mathcal{M}^{\frac{2}{3}}\rho^{\frac{1}{3}}\beta \\ 1-\beta &\propto \mathcal{M}^2\beta^4 \end{aligned}\right\} \tag{38}$$

where T_e and T_c are the effective and central temperatures, respectively, g the acceleration due to gravity, and $1-\beta$ the ratio of radiation pressure to gravitational attraction. Assuming $\mathcal{M} \propto R$ we find then

$$\left.\begin{aligned} &\mathcal{M}^2\rho = c_1;\ \mathcal{M}g = c_2;\ Rg \propto c_3;\ \rho \propto g^2 \\ &T_c \propto \beta;\ T_e \propto T_c\mathcal{M}^{\frac{1}{4}} \propto \mathcal{M}^{\frac{1}{4}}\beta \propto T_c\rho^{-\frac{1}{8}} \end{aligned}\right\} \tag{39}$$

where c_1, c_2, and c_3 are constants. Further, with the aid of (35)

$$\mathcal{M} \propto (\gamma a)^{\frac{1}{2}}P;\ T_e \propto T_c(\gamma a)^{\frac{1}{8}}P^{\frac{1}{4}} \propto \beta(\gamma a)^{\frac{1}{8}}P^{\frac{1}{4}}. \tag{40}$$

The relation $T_e \propto \mathcal{M}^{\frac{1}{4}}\beta$ is of special interest. Since β diminishes faster than $\mathcal{M}^{\frac{1}{4}}$ increases, T_e must decrease with increasing $\mathcal{M}$. The change for the range in $\mathcal{M}$ shown above corresponds to an increase of about one spectral interval, which agrees well with the observed range in spectral type. At the same time M decreases from -1.2 to -5.1.

The formulae of radiative equilibrium admit of no dispersion in the case of individual stars. This is also true of the relations (39) and (40) for the Cepheids. In the case of the masses and densities, we have already found reasons for believing that this is not strictly in accordance with the facts. Although the dispersion is small, it is in part undoubtedly real, for at present the theory necessarily neglects the modifications of gravitational acceleration

[1] Eddington, *Monthly Notices*, **79**, 180, 1919.

produced by the rotation of the stars on their axes, which must influence the relation of radiation to other stellar characteristics.

16. RELATION TO RUSSELL'S FREQUENCY DIAGRAM—REMARKS ON THE DISTRIBUTION OF MASS AND DENSITY

The justification for the discussion of Eddington's theory of Cepheids in this place is the light thrown upon the distribution of mass and density. Equations (35) and (37), for example, afford a basis for a further revision of the mass- and density-curves; but an additional approximation seems scarcely justified at present. Table XVII shows already a surprisingly good agreement between his results and those in Figure 3. Further modification would affect mainly the stars of masses 10 or larger, and these are not numerous. Moreover, Eddington himself has repeatedly emphasized the fact that his theory is but a first approximation, in that certain simplifying assumptions had to be introduced in order to make a beginning. Figure 3 is therefore allowed to stand as it is for the present, although the data in Table XIX are added to show,

TABLE XIX

MASS, ABSOLUTE MAGNITUDE, AND SPECTRUM, EDDINGTON'S THEORY OF CEPHEIDS

$\mathfrak{M}$	M	Sp.	$\mathfrak{M}$	M	Sp.
2	−0.9	F1.0	10	−3.4	F8.0
3	−1.4	F4.0	12	−3.6	F8.4
4	−1.9	F5.2	15	−4.0	F9.0
6	−2.5	F6.5	20	−4.6	F9.3
8	−3.0	F7.5	25	−5.1	F9.6

The values of the masses are adjusted to give a mean value equal to that derived from Figure 3.

in a general way, what would result, for the Cepheids at least, from rigorous application of the theory. The values of M and the spectrum may be regarded as co-ordinates for the location of corresponding values of $\mathfrak{M}$ in a diagram similar to Figure 3.

It is, of course, not to be supposed that the stars are uniformly distributed along the mass-curves of Figure 3. In general they cluster about certain lines of maximum frequency which, with one exception, bear little relation to the lines of equal mass. The most conspicuous of the frequency lines is that defined by the dwarfs,

which continues through the A's and joins smoothly with the line of maximum frequency of the B stars. Along this line we have found a regular decrease in the average mass with increasing spectral type and absolute magnitude.

The number of A stars of very high luminosity seems to be relatively small.[1] If this is really the case, there must be a region of low frequency in the diagram between the most luminous B's and the very bright giants of the later types. Beyond Fo the giants fall into two groups, the Cepheids and pseudo-Cepheids, and a large group of G, K, and M stars whose absolute magnitudes lie mainly between $M=-0.5$ and $+2.5$. The Cepheids and pseudo-Cepheids attain very high luminosities and show a rather definite correlation line running approximately from $M=-0.4$, F4, to $M=-4$, G8, which in general cuts the mass lines obliquely. For the Cepheids alone, its course is determined by the intersections of the mass lines of Figure 3 with those defined by equation (37), which, empirically at least, is a condition of Cepheid variability.

The frequency line for the second and more numerous group of giants seems, on the other hand, to be definitely related to the distribution of mass. It has a clearly marked minimum at Ko and coincides approximately[2] with the mass line $\mathcal{M}=2$ as shown in Figure 3. The chief difference is a steeper descent from Go to Ko and an even more pronounced minimum than is shown by the mass line. The coincidence is probably significant, for Eddington[3] finds on theoretical grounds that, among the giants, masses of this order of magnitude may be expected to have the highest frequency.

[1] The frequency diagram of Russell for the helium stars of Kapteyn and the stars in the list of spectroscopic parallaxes is illustrated in Annual Report, Mt. Wilson Observatory, *Year Book Carnegie Institution of Washington*, 1921.

[2] The position of this correlation line and of the line of maximum frequency for the G, K, and M giants is shown in Fig. 1. This figure also shows the course of the equal-mass line $\mathcal{M}=2$. The correlation of spectral type with absolute magnitude for the most luminous stars of our own immediate system is essentially that found by Shapley for the brightest stars in globular clusters; and it now appears from results not yet published that in some of the galactic clouds a similar correlation with apparent magnitude is to be found. It should be possible in such cases to estimate roughly the distances of the clouds. Provisional results indicate values of the order of 20,000 to 50,000 light years. See *Annual Report, loc. cit.*

[3] *Report of the British Association for the Advancement of Science*, 1920, p. 43.

These circumstances seem to afford justification for the peculiar course of the mass lines for the late-type giants, which otherwise must depend solely upon the applicability of the principle of equipartition.

The minimum itself, or what remains of it after the reduction of the visual absolute magnitudes to bolometric values in the manner indicated above, must not be taken too seriously, however, for the residual irregularity in the mass lines is little in excess of the small systematic uncertainties affecting the absolute magnitudes of the different spectral types.[1]

In this connection the result of page 183 for the masses of fourteen giant visual binaries may be recalled. Divided by 1.75 to reduce to the central component, the mean mass becomes 1.7 ± 0.9. For the same stars we find by interpolation from Figure 3, with spectral type and absolute magnitude as arguments, the mean value $\mathcal{M} = 2.7 \pm 0.2$ The agreement is not good, but falls within the limits of uncertainty incident upon the small number of stars available for the comparison.

The decrease in mass along the dwarf branch from $\mathcal{M} = 10$ at Bo to $\mathcal{M} = 0.6$ at Ma raises a question of much interest, especially when it is recalled that the dispersion in mass is certainly small.[2] Practically, large masses are not to be found in our lists of dwarf stars. And yet the large-mass early-type stars are numerous. Why, therefore, do we not find more large masses among the dwarfs?

Several possibilities must be considered. Thus we might assume that the stars now on the dwarf branch (including the B's, as usual) began their development at much the same time, and that those of largest mass have only partially run their course and are now in the B stage. The assumption would seem less arbitrary could we suppose that all the stars began their evolutionary careers at the same time; but this apparently is not the case, for numerous objects with only moderate masses of 2 or 3 are still in an early stage of development. The most serious objection to this explanation, however, is that it takes no account of a peculiar kind of selection.

[1] Strömberg, *Mt. Wilson Contr.*, No. 220; *Astrophysical Journal*, **55**, 11, 1922.

[2] Russell has also discussed this correlation., *Observatory*, **37**, 173, 1914.

Let us therefore make the contrary assumption and suppose that stellar development has continued so long that all possible masses really occur among the dwarfs of late type. Among these, the large masses, say of the order of 10, will be of very low frequency as compared with the modal value for all types together. Kapteyn and van Rhijn place the maximum of the frequency-curve of absolute magnitudes at $M=+7.7$. The modal value of the mass therefore should be about 0.7, and since the dispersion is small, the large masses must be very infrequent.

Consider now only the stars in our lists of data. The dwarfs among these will all be included in a restricted region of space near the sun. The small masses will greatly predominate and the mean mass will be low. The giants, on the other hand, will be scattered through a large volume of space, and their average mass will be large, because only stars of large mass can attain high luminosity. The B's among them will form the collection at the upper end of the dwarf branch whose mean mass we have found to be 10. The total number of these in our catalogues will be considerable, because they represent those present in a large volume, whereas the large masses among the dwarfs will be absolutely as well as relatively infrequent, because we can see only those which are near us.

The stars of intermediate luminosity fill in the gap, and we must therefore expect to find a correlation of mass with absolute magnitude and spectral type very similar to that actually observed. The average mass decreases with advancing type; at the same time the number of stars of large mass in our catalogues decreases rapidly and the smaller masses appear in succession and in increasing numbers because they can attain the temperatures corresponding to the later spectral types.

One other possibility should also be borne in mind, namely, a decrease in mass through loss of energy by radiation. From the standpoint of relativity[1] any change dE in the total energy of a system implies a corresponding change of dE/c^2 in the inertial mass, where c is the velocity of light. A similar conclusion follows

[1] Einstein, *Annalen der Physik*, **18**, 639, 1905.

also on the basis of the Newtonian mechanics[1] from the phenomenon of radiation pressure, for, unless we attribute mass to radiation, there are certain cases, at least, in which neither the energy nor the motion of the center of gravity will be conserved. Since the principles of mechanics are inductions, we cannot be sure that they apply to physical phenomena with the degree of precision implied by assuming that mass actually changes with changes in the energy. But considerations of consistency are sufficient to give the matter importance, and it derives added interest from the fact that a decrease in mass has been invoked as a means of accounting for the enormous unknown supply of energy which is now generally recognized as an essential feature of stellar phenomena. The suggestion that the packing of electrons and the nuclei of hydrogen atoms to form nuclei of heavier elements, with an accompanying loss in mass which reappears in the form of great quantities of radiant energy, attaches itself naturally to the relation between mass and energy that harmonizes with fundamental mechanical principles, whether of Einstein or of Newton.

Of these three factors which may affect the correlation of mass and spectral type, selection certainly enters in the manner indicated, but its quantitative evaluation is at present impossible. This leaves the operation of the other two factors entirely speculative. If, however, there is any appreciable change in the mass of the stars during their development, they will not follow the mass lines of Figure 3, but cut across them obliquely; and this leads to a peculiarity of these lines that immediately attracts attention.

The equal-mass lines do not meet the dwarf branch tangentially, as one might expect on the basis of a simple gravitational contraction with its accompanying changes of temperature, but intersect the frequency line at large angles. A comparison of Figures 2 and 3 shows that in this particular there is a large element of uncertainty; and yet the evidence seems to exclude the possibility of tangential intersections, unless we are prepared to abandon the principle of equipartition at the point where it seems to be best justified, namely, in the vicinity of the dwarf branch.

[1] *Ibid.*, **20,** 627, 1906. This result seems also to have been pointed out by others—Planck, for one, I believe—although I have been unable to locate the source.

The space-velocities of stars of any given type, say F5, show a definite correlation with absolute magnitude, the mean velocity increasing continuously as the brightness decreases. If equipartition holds for the stars of a given type which are near the frequency line of the dwarfs, as it does in the mean for the stars of different types scattered along the frequency line, there must be a continuous decrease in mass downward along the vertical lines of Figure 3. The data of Table X give a numerical expression to this relationship of mass to luminosity, which, quantitatively, should be most exact for values of M near the frequency line of the dwarfs. Now on this line, for spectral type F5, as an illustration, we find a mean mass of 1.5. Above this point in the diagram, that is, for more luminous stars, the masses are larger; below they are smaller. If the decrease in mass downward is really continuous, it follows at once, since the mass also decreases continuously along the dwarf frequency line, that curves connecting equal masses cannot meet the frequency line tangentially, but must intersect it at a considerable angle. In fact Figure 2 shows that the strict application of the principle of equipartition leads to angles in excess of 90°, counting as 0° the tangential junction required by gravitational contraction. The evolutionary path of any given star, however, when once it has joined the dwarf branch, must lie in the general direction of the frequency line.

The explanation of the contradiction lies in the peculiar selection which determines the decrease in mass along the dwarf branch. Consider for example two dwarf stars of Type F5, one a large mass which in its development has followed the dwarf branch downward from some earlier type, the other a star of mass so small that it has been unable to attain the high temperatures of the very early types and joins the dwarf branch at Type F5. The star of large mass will be the brighter of the two. Hence the path of the small star must intersect the line of maximum frequency at some point a little in advance of F5 in order that it may swing into position below the larger star. The path of the larger star, on the other hand, will lie above the frequency line. Similar reasoning applies to stars of intermediate mass and it is evident that the entire group of F5 dwarfs will show a correlation of mass

with absolute magnitude similar to that found from the principle of equipartition. In their further evolution these stars will move as a group along paths approximately parallel to the line of maximum frequency. But these paths do not and cannot coincide with the equal mass lines of Figure 3, for when the group in question is at F5, there is at G0 another group, whose luminosities show a similar correlation with mass, but whose masses, because of selection, are systematically smaller than those at F5. If the points in the diagram corresponding to equal masses in these two simultaneously observed groups be connected, the lines of equal mass will cut the frequency line at high angles as shown in Figure 3. Hence there is no conflict between the mass-distribution shown by this diagram and the gravitational theory of stellar development. The only point is that in the vicinity of the dwarf branch the stars do not follow the equal mass lines.

17. IONIZATION AND MEAN ATOMIC WEIGHT

The final equations of Eddington's theory[1] are

$$L=\frac{4\pi cG\mathfrak{M}}{k}(1-\beta) \tag{41}$$

$$1-\beta=0.0026\ \mathfrak{M}^2\beta^4m^4 \tag{42}$$

where c and G are constants. These formulae refer to the giant stage and assume that the mean atomic weight is independent of the temperature. Variations in m during the star's development would affect the relation between luminosity and spectral type. The absolute value of m fixes the relation between mass and radiation-pressure β, and hence determines the values of the mass which are most likely to occur.

It is unnecessary to repeat here Eddington's arguments in favor of a small value of m. In general these seem to be only strengthened by the results of recent investigations of ionization phenomena. Eggert,[2] for example, has considered the behavior of the iron atom when subjected to pressures and temperatures comparable with those at the center of a typical giant star and finds that 16 of the

[1] *Astrophysical Journal*, **48**, 208, 210, 1918.

[2] *Physikalische Zeitschrift*, **20**, 570, 1919.

26 electrons would be stripped from the atom, thus reducing the mean "atomic" weight to 3.3.

It is easy to extend this result to other parts of the star. Using the reaction isobar of Nernst and basing the calculation of the energy of dissociation on the dynamical relations of the Bohr atom, Eggert finds as the condition for the removal of 8 electrons from 99 per cent of the atoms[1]

$$2.68+0.4 \log P=\frac{-2\times 10^5}{T_i}+\log T_i \tag{43}$$

in which the pressure P is measured in atmospheres. For the separation of 16 electrons the corresponding condition is approximately the same as (43) except for the appearance of an additional factor of 10 in the first term on the right.

Equation (43) and its alternate may be used to calculate the values of T_i corresponding to the pressures at different points along the radius. The comparison of the ionization temperatures thus found with the corresponding stellar temperatures should then give some indication as to the variation in ionization throughout the star.

The total pressure P is given by the ordinary gas equation, divided by β, the factor arising from radiation-pressure.

$$P=\frac{p}{\beta}=\frac{\mathfrak{R}}{\beta m}\rho T \tag{44}$$

T and ρ may be found from Emden's[2] formulae, or we may use the values given by Eddington[3] for his typical star ($\mathcal{M}=1.5$, $\rho_m=0.002$), taking care to modify those for T, which contain the factor m, to make them conform with the degree of ionization expressed by (43) and its alternate. Since T contains both m

[1] Professor Paul Epstein of the California Institute of Technology calls my attention to an error in Eggert's expression for K (*op. cit.*, p. 573). A factor 8^8 has been omitted. It has little effect, however, on equation (43), the absolute term being increased from 2.32 to 2.68.

[2] *Gaskugeln*, p. 97, 1907. Eddington has called attention to the misprint in the formula for θ^3. The last factor in the denominator should be squared.

[3] *Astrophysical Journal*, **48**, 213, 1918. Unless I am in error, Eddington's values for the temperature have been computed with $m=2$, whereas the text seems to imply that 2.8 has been used. If so, the tabular values should be increased by 40 per cent.

and β as factors, the total pressure P depends upon neither the value of the radiation-pressure nor the degree of ionization. T, however, depends upon both, and, moreover, β also depends upon m (see equation [42]).

For the removal of 8 and 16 electrons, respectively, the values of m become 6.2 and 3.3. Corresponding to these, for $\mathcal{M}=1.5$, $\beta=0.496$ and 0.766. With these data we find the values of P and T given in the second, third, and fifth columns of Table XX

TABLE XX

IONIZATION AND STELLAR TEMPERATURES, IRON IN EDDINGTON'S TYPICAL STAR

r	Log P (Atm)	−8 Electrons $m=6.2$		−16 Electrons $m=3.3$		Percentage of Mass Outside r
		T	T_i	T	T_i	
0	7.40	8.7×10^6	8.0×10^5	7.1×10^6	2.6×10^6	100
1	7.13	7.4	6.7	6.1	2.4	88
2	6.47	5.0	4.9	4.1	2.0	48
3	5.63	3.1	3.4	2.5	1.6	18
4	4.69	1.8	2.5	1.5	1.3	5
5	3.59	0.96	1.8	0.79	1.1	0.7
6	1.98	0.39	1.2	0.32	0.8	0.005

Radius of star = 6.9, $\mathcal{M}=1.5$, $\rho_m=0.002$.

for various points along the radius. The unit for r is such that the distance from the center to the surface is 6.9. Substituting the values of P into Eggert's formula, we find the ionization temperatures given in the fourth and sixth columns of the table. These temperatures, required for the removal of 8 and 16 electrons, respectively, may be compared with the corresponding stellar temperatures. The latter, which change little with the assumed change in m, are approximately ten times the temperature required to separate 8 electrons from the nucleus, and of the same order as the ionization temperatures corresponding to the removal of 16 electrons. The decrease in pressure toward the surface compensates for the fall in temperature to such an extent that the ionization should be nearly constant throughout the star. It is only within a thin shell near the surface, which contains an insignificant fraction of the mass (see last column, Table XX), that m can rise much above the value 3.3.

Eggert's formula is an extrapolation and involves assumptions which cannot at present be directly controlled; but its general indications are in close agreement with Eddington's conclusions. Even admitting the various underlying assumptions, the result applies only to a star consisting wholly of iron; but one can scarcely doubt that the mean values of m for all the elements actually present must be small and nearly constant for all points within the star except those close to the surface.

It seems worth while also to consider the relation of ionization to stellar temperature in giant stars of constant mass but differing spectral types, in order to gain some idea of the permanence of this relation as a star contracts under the action of gravitation.

Consider stars of Eddington's critical mass $\mathcal{M}=2$, which, as we have seen, are those occurring most frequently among the late-type giants. It will be sufficient to calculate the central temperatures and pressures, and in view of the preceding results it is clear that we may use $m=3.3$. The corresponding value of β is 0.70. Using Emden's formula as before (remembering that for the type of equilibrium considered $\rho_c=54.25\ \rho_m$), we find the results for T_c and P_c given in Table XXI. The values of M and

TABLE XXI

CENTRAL TEMPERATURES AND PRESSURES, AND IONIZATION TEMPERATURES FOR IRON ($\mathcal{M}=2$, $m=3.3$)

Sp.	M	Log ρ_m	Log T_c	Log P_c (Atm)	T_c	T_i
Ma	0.0	−5.1	6.10	4.3	1.3×10^6	1.2×10^6
K5	+0.7	−4.1	6.43	5.6	2.7	1.5
K0	+1.1	−3.1	6.77	7.0	5.9	2.1
G5	+0.8	−2.7	6.90	7.5	7.9	2.6
G0	+0.7	−2.2	7.07	8.2	11.7	3.3
F5	+2.0	−1.0	7.47	9.8	29.5	7.0

log ρ_m in the second and third columns were read from the diagram of Figure 3. The ionization temperatures for the removal of 16 electrons are in the last column of the table.

These data represent the sequence of changes from an early stage of development, on the basis of Russell's theory, through to the dwarf stage, where the density is so large that Boyle's law is no

longer accurately obeyed. The increases in T_c, ρ and P_c are approximately 20, 10,000, and 300,000 fold, respectively. In spite of these extraordinary changes, the ionization temperatures remain of the same general order as those at the center of the star throughout its development. Initially, the two temperatures are sensibly equal, and the gain in T_c over T_i is so slight that any further decrease in m must be small. The complete removal of the third sheath of electrons, for example, would give $m=2.2$.

Even for stars of the lowest density, such as Antares or Betelgeuse, conditions are very similar to those illustrated by the first line of Table XXI. Low density means relatively low central temperature and pressure, but this factor is compensated by the large masses which characterize the very luminous giants, so that the relation of ionization to stellar temperatures is much the same as for the objects already examined.

It may be remarked incidentally that the values of T_c in Table XXI throw some light upon the behavior of the effective radiating layers in stars of different types. The increase in T_c is approximately ten times that in the effective temperature. But for the type of equilibrium in question the temperature gradient is independent of the density, that is, the temperature increase for homologous points along the radius should be constant. The explanation must be that the thickness of the radiating layer in an Ma giant, because of the enormous difference in density, is very much greater than that for an F5 dwarf. As development progresses, radiation from the deeper and hotter strata is absorbed as a result of increasing density, and, in consequence, the increase in effective temperature is less rapid than would otherwise be the case.

On the basis of the preceding evidence we must expect a high degree of ionization at the earliest stages of stellar development with which we are familiar, and comparatively little decrease in the value of m as the development proceeds. The Bohr atom affords the only foundation we have for estimates such as have been made, and it is not certain that the results are of the right order of magnitude. One consequence of their acceptance, however, should not be overlooked, and that is the total amount of

energy required for ionization. For the removal of 8 electrons this equals 1.8×10^{7} calories per gram atom,[1] corresponding to an ionizing potential of about 800 volts. The mass of Eddington's typical star is approximately 2.9×10^{33} grams. The removal of 8 electrons from all the atoms in this mass would therefore require

$$U_o=\frac{2.9\times10^{33}\times1.8\times10^{7}\times4.2\times10^{7}}{56}=4\times10^{46}\text{ ergs,}\tag{45}$$

while for the removal of the second sheath of 8 electrons a quantity of the order of ten times this amount, or

$$U_o=4\times10^{47}\text{ ergs}\tag{46}$$

would be necessary.

Now the total energy generated by gravitational contraction of the star from a state of infinite diffusion is[2]

$$\Omega=\frac{3(\gamma-1)}{5\gamma-6}\frac{G\mathcal{M}^2}{R}\tag{47}$$

where γ is the ratio of specific heats, G the gravitational constant, 6.66×10^{-8} cm³/grams sec.², and R the radius, 7×10^{11} cm; γ lies between the limits 4/3 and 5/3, whence the fractional coefficient on the right has the limiting values 3/2 and 6/7. At a maximum, therefore, which also corresponds to Eddington's model,

$$\Omega=\tfrac{3}{2}\cdot\tfrac{2}{3}\cdot\frac{10^{-7}\times9\times10^{66}}{7\times10^{11}}=1.3\times10^{48}\text{ ergs.}\tag{48}$$

Of this total the etherial energy alone,[3] namely,

$$(1-\beta)\Omega=0.234\,\Omega=3.0\times10^{47}\text{ ergs,}\tag{49}$$

[1] Eggert, *loc. cit.*

[2] Emden, *Gaskugeln*, p. 125, 1907.

[3] Eddington, *Monthly Notices*, **79**, 23, 1918. Anderson, with whom I have had much stimulating discussion during the preparation of this paper, calls my attention to the fact that ionization cannot be produced by the collision of electrons with atoms without violating the Nernst formula of equilibrium, which is well established on thermodynamical grounds. This equation depends upon the pressure, whereas the hypothesis of ionization by collisions alone leads to a condition for equilibrium which is independent of pressure. If the effective energy is assumed to be partly radiant and partly kinetic, the pressure would enter, but not in the manner required by Nernst's equation. Since the internal kinetic energy of the atoms is not available, the radiant energy alone can be operative (as in the case of the photoelectric effect, for example). See also Milne, *Observatory*, **44**, 269, 1921.

seems to be available for ionization. This is of the same order as U_o, the energy of dissociation given in (46). Admitting, therefore, the validity of Bohr's theory and the applicability of the Nernst formula of equilibrium, the insufficiency of the gravitational theory presents itself in a new form. As an alternative, the question is perhaps raised as to whether the giant stars, even in their very earliest stages of development, are ever in a state other than one of high dissociation.

The calculation of the stellar temperatures in Tables XX and XXI neglects altogether the depletion of the available supply of energy by ionization. Were gravitation the only source, we could expect neither the temperatures nor the degree of ionization indicated by the discussion; but, instead, an equilibrium state which would be established for something less than the extremes of temperature and ionization shown by the tables.

18. COMPARISONS WITH EDDINGTON'S FORMULAE

The assumption concerning m which underlies equations (41) and (42) has been discussed in the preceding section. These formulae also assume that the outflow of energy across any surface of unit area is proportional to the gravitational acceleration at that point ($H \propto g$), and that the mass coefficient of absorption, k, is a constant. The energy assumption can also be stated in the form that the outflow throughout the star per unit of mass is constant. The source of the energy is not specified; it may be derived from gravitational contraction or any other source capable of providing an adequate supply. Assuming further that the transfer of energy is by radiation and not by convection, the conditions for equilibrium are found to be the same as those for adiabatic equilibrium with a ratio of specific heats equal to 4/3. It then follows through (41) and (42) that the bolometric magnitude of a star remains constant so long as it behaves as a perfect gas.[1]

A giant star in its development, however, presumably follows closely one of the equal-mass lines in the upper part of Figure 1, for

[1] As Jeans points out, *Monthly Notices*, **79**, 319, 1919, this general result follows at once, by simple considerations, from the assumption H/g=const. The nature of the dependence of radiation on mass and other characteristics is, however, another matter, less easily disposed of.

the selection which enters along the dwarf branch has comparatively little influence on the average mass of the giants which come under observation. Observations indicate, therefore, that the absolute magnitude is not independent of the stage of development. The change is even larger than shown in Figure 3, for, as already stated, the visual absolute magnitudes must be corrected by the quantities given on page 194 to make them comparable with the bolometric magnitudes calculated by (41).

The question now arises as to the particulars in which the underlying assumptions must be modified in order to effect a closer agreement with observations. Such consideration as can be given here may be prefaced by the remark that Jeans,[1] starting from the assumptions of radiative conduction, gravitational contraction, and constant k and m, finds that the radiation of a star will remain constant as long as Boyle's law is obeyed. This is also the case if the star has sources of energy other than gravitational contraction, provided the rate of generation per unit mass is independent of the time. This is Eddington's result without his assumption that $H \propto g$; but the functional dependence of radiation upon the mass cannot be specified. Jeans also finds that H/g will be constant throughout the mass of gas only when the mass is so large that radiation-pressure predominates and β may be put equal to zero. For the ordinary run of stellar masses, however, H/g will not be constant within the star, but for a given point will be independent of the time, provided the only source of energy is gravitational.

Eddington[2] himself points out that H/g cannot be rigorously constant within the star, for, assuming gravitational contraction through a series of homologous states and a temperature- and pressure-distribution corresponding to adiabatic equilibrium with $\gamma = 4/3$, the outflow of energy per unit mass would increase about 70 per cent from the surface to the center.

In view of a probable variation of H/g of the order indicated, the original assumption gives close agreement with observations. The matter can be tested by comparing the calculated values of M corresponding to different values of $\mathcal{M}$ with that shown by Figure 3

[1] *Loc. cit.*

[2] *Monthly Notices*, **77**, 599, 1917.

for stars of the same spectral type. By proceeding in this manner we avoid disturbances arising from variations in k and m, since for stars of the same type these quantities should be less subject to change.

The results are shown in Table XXII. This table gives the deviations of the observed values of M (Fig. 3) from those calculated

TABLE XXII

Observed and Calculated (Eddington) Variation of Absolute Magnitude with Mass

$\mathcal{M}$	Cal. M	O−C				$\mathcal{M}$	Cal. M	O−C			
		Fo	Go	Ko	Ma			Fo	Go	Ko	Ma
1.5	−0.3	.....	+1.5	+0.7	+0.3	6	−2.9	+0.3	−0.3	−0.1	0.0
2	−0.9	.....	+0.9	+0.3	+0.2	7	−3.1	−0.1	−0.5	−0.1	0.0
3	−1.7	.....	+0.5	0.0	0.0	8	−3.3	−0.5	−0.6	−0.2	0.0
4	−2.2	+1.4	+0.1	0.0	0.0	9	−3.5	−0.7	−0.8	−0.3	0.0
5	−2.6	+0.7	−0.1	0.0	0.0	10	−3.7	−1.0	−0.9	−0.3	0.0

by Eddington, which are in the second column of the table. Zero point corrections have been applied to the latter to make the mean O−C for each type equal to zero. There is a large progressive change for the F stars, but the agreement improves with advancing type and, with the exception of one difference, is practically perfect for the K and M stars. From this comparison it is clear that the error in the assumption $H/g=$const. is not very serious when we consider stars of a specified type.

The lack of parallelism of the equal-mass lines with the axis $M=0$ is difficult to account for because the assumption of constancy for both k and m enters here, as well as that concerning the flow of energy. In general the luminosity for any given mass decreases with increasing temperature. For mass 4, as an illustration, the bolometric absolute magnitude increases from -3.1 at Ma to $+0.3$ at Fo. This is in the direction corresponding to decreasing values of m, and one might expect much of the change to be accounted for by increasing ionization. The evidence of section 17, however, is against any considerable variation in m. Even in the earliest visible stages of development we should not anticipate a value

much in excess of 3.3, and unless there is a large amount of nuclear disintegration, which seems improbable, the value of m cannot fall below 2. It is easily shown that a decrease from $4\sqrt{2}$ to 2 would cause an increase in M of about one magnitude, which is only a third of the amount required. The admission of much larger initial values of m is insufficient to bridge the gap, for (42) shows that $1-\beta$, and hence the luminosity, are not much affected as m is increased.

The behavior of k is obscure. Although the conditions within the star which determine the degree of absorption must be analogous to those governing the absorption of X-rays, it should be borne in mind that the effective value of k is that corresponding to the outermost layers of the star where the temperature is very much lower than in the interior. Even near the surface the opacity is large, but we have no data as to its variations that are certainly applicable.

This leaves the larger part of the increase in the absolute magnitude unaccounted for, and it is not clear as to how the discrepancy is to be explained. If the contraction is homologous and the energy gravitational, or gravitational plus a source independent of the time, we should not expect variations in outflow with advancing development other than those caused by changes in k and in m.

As an explanation, changes in m that would be sufficient seem to be excluded; those of k are in doubt. There remain, however, the assumptions underlying the outflow of energy. The source of energy almost certainly is not wholly gravitational, and if not, with almost equal certainty, is not independent of the time. The outstanding questions therefore lead directly into the realm of atomic physics, and it seems more than ever likely that the problem of the stars will find its solution in the solution of the problem of the atom.

In conclusion, attention may be directed to the relation of mass to luminosity along the dwarf branch. Here Boyle's law is in general no longer obeyed. It will be noted, however, that the density increases slowly until the late types are reached, which suggests that the departure from the conditions of a perfect gas

may affect the behavior of stars of different spectral types much alike, so that a formula giving the relation of luminosity to mass for the giants, with the exception of a constant, would also hold approximately for the dwarfs.

The matter is easily tested by calculating $\mathcal{M}$ from (41) for the values of L corresponding to the absolute magnitudes given in the second column of Table IV. Choosing the constant factor so that the mean of the values thus calculated agrees with the observed mean mass, we have the results in Table XXIII. With the

TABLE XXIII

VALUES OF MASS ON DWARF BRANCH

Sp.	Obs.	Cal.	Sp.	Obs.	Cal.
B0	10	18	G0	1.0	1.0
B5	8.3	10.5	G5	0.76	0.76
A0	6.0	6.6	K0	0.68	0.62
A5	4.0	4.1	K5	0.62	0.48
F0	2.5	2.5	Ma	0.59	0.26
F5	1.5	1.5			

exception of a spectral interval at either end of the series, where the deviations from the average density are large, the agreement is good.

I must express my grateful acknowledgments to Miss Joyner and Miss Richmond, of the Computing Division, for their efficient assistance in the extensive tabulations and calculations involved in the preceding discussion.

MOUNT WILSON OBSERVATORY
December 1921

ON THE CALCULATION OF MASSES FROM SPECTROSCOPIC PARALLAXES[1]

By HENRY NORRIS RUSSELL[2]

ABSTRACT

Calculation of masses of stars from spectroscopic parallaxes.—It appears probable that the line intensities upon which spectroscopic parallaxes are based are *functions of the temperature and density of the star's atmosphere.* If this were exactly true, all stars of the same surface brightness and density would have the same spectroscopic absolute magnitude, and *the masses, computed from the spectroscopic parallaxes, would come out the same for all the stars* of such a group (whatever the dispersion among their actual masses) and equal to the geometrical mean of the latter. To obtain a reliable measure of the dispersion in mass among binary stars, parallaxes must be determined in some other way.

Spectroscopic and dynamical parallaxes.—It follows that the spectroscopic parallaxes and the dynamical parallaxes (derived on the assumption that the mass of a binary system is equal to the mean value for stars of its spectral type and absolute magnitude) are *systematically equivalent to one another*, and really rest on the same physical relationships and assumptions.

Dispersion of mass among visual binaries (dwarf stars).—By an indirect method, depending on Strömberg's comparisons of spectroscopic and trigonometric parallaxes, Seares's conclusion that the dispersion is small is confirmed. The probable error of dispersion of log $\mathfrak{M}$ appears to be less than ± 0.2, but cannot be exactly determined.

The very small values obtained by Seares in the preceding paper[3] for the dispersion in mass among dwarf stars suggest the following explanation.

It appears very probable, from physical considerations, that the spectral type of a star is determined by the temperature T of its outer atmosphere, while the characteristics associated with the absolute magnitude depend also, and mainly, upon the density of the atmosphere, ρ'. If M_s is the absolute magnitude, determined spectroscopically, we may then write

$$M_s = F(T, \rho'). \qquad (1)$$

It is also probable that the surface brightness j depends almost entirely on the temperature, and that the density of the atmosphere

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[3] *Mt. Wilson Contr.*, No. 226.

above the visible surface is a function of the mean density ρ (though not necessarily proportional to it). We have then

$$M_s = f(j, \rho). \tag{2}$$

It may be that M_s depends to some degree on other variables, but it is probable that their influence is slight. The form of the function f cannot be predicted by theory, but can be found empirically when sufficient data regarding stellar masses and densities are available.

Assuming the relation for the moment to be exact, and introducing it into equation (22) of Seares's paper, we have

$$\log \mathcal{M}_s = \log \rho + 0.6j - 0.6f(j, \rho) + 2.77. \tag{3}$$

Hence $\mathcal{M}_s$ the mass corresponding to the spectroscopic parallax is itself a function of ρ and j alone. Like (22) this is a general result, holding good no matter how the mass is calculated, so long as the formula involves the absolute magnitude and our assumptions about the latter are correct.

Stated in words, this signifies: *If the spectroscopic absolute magnitude of a star depends only on its surface brightness and density, the masses computed from the spectroscopic parallaxes will be identical for all stars which have the same density and surface brightness, no matter how different their actual masses may be.*

In still other words, the spectroscopic parallax (on our assumptions) will be correct only in the case of stars having a certain average value of the mass. For larger or smaller masses there will be a systematic error in the absolute magnitudes, making the former too faint and the latter too bright, which will alter the spectroscopic parallax to just such a degree as to conceal entirely the real differences in mass.

If the spectroscopic calibration curves are adjusted so as to give a correct value for the mean absolute magnitude of the stars, the "spectroscopic mass" will be the geometrical mean of the individual masses, since, if j and ρ are constant, M varies proportionally to $\log \mathcal{M}$.

To determine the dispersion in mass among visual binaries, it is therefore necessary to have recourse in some way to parallaxes

other than spectroscopic. One way of doing this is through Strömberg's[1] discussion of the errors of the spectroscopic parallaxes—which he has determined by comparison of individual spectroscopic and trigonometric parallaxes, allowing for the errors of the latter. Let the probable error thus found for log π be $\pm r$. Three sources contributory to this error may be distinguished: (1) the error just discussed, arising from differences between the masses of individual stars of the same surface brightness and density; (2) the error arising from other physical causes, on account of which the functional relation (2) is not exactly true; and (3) errors of observation. If we call the amounts of these, expressed as probable errors of log π, respectively, r_m, r_p, and r_e, we will have

$$r^2 = r_m^2 + r_p^2 + r_e^2 .$$

If, on the other hand, we compute individual masses as Seares has done, the error r_m will be without influence on these masses, while r_p and r_e will affect log $\mathcal{M}$ by 0.6 of their amounts, and there will be an additional source of error, due to the imperfections of the orbits and also to errors in the assumed ratio of the masses of the components, which we may call $\pm r_c$. If then r' is the probable error of distribution, which Seares calls $r(\Delta \log \mu)$, we have

$$r_m^2 + r_p^2 + r_e^2 = r^2 = (0.36)^2$$

$$0.36(r_p^2 + r_e^2) + r_c^2 = r'^2 = (0.22)^2$$

$$r_m^2 = 2.78\, r_c^2 - 0.004 .$$

This would indicate that the actual dispersion in log $\mathcal{M}$ (which corresponds to the probable error $\pm 0.6\, r_m$) is very small—being substantially equal to the spurious dispersion which arises from errors in the orbital elements of the binaries.

The observed values of r and r' are, however, subject to some uncertainty, and it is possible that some other concealed correlation may still be making the agreement appear better than it should.

It is obvious, however, that a better way of determining the dispersion in mass among visual binaries of the same spectral type will be a direct comparison of "hypothetical" or dynamical paral-

[1] *Mt. Wilson Contr.*, No. 199, p. 15; *Astrophysical Journal*, **53**, 13, 1921.

laxes, computed with a mean value of the mass, with trigonometric parallaxes. In this case there seems to be no reason to fear a correlation of the sort discussed above.

It may be noticed that the dynamical parallax of a binary, calculated on the assumption that the principal star has the mass $\mathcal{M}_s$ derived from spectroscopic parallaxes of stars of similar kind, and the spectroscopic parallax itself, must agree exactly in every case (barring the effects of errors of observation and the small uncertainty arising in the estimate of the ratio of the mass of the system to that of the brighter component), so that it may fairly be said that spectroscopic and dynamical parallaxes are systematically equivalent to one another, so long as the mass used in computing the dynamical parallaxes is taken as a function of both absolute magnitude and spectral type. The two then rest essentially on the same physical assumptions—though this is far from obvious at first sight. If this conclusion is correct, spectroscopic parallaxes are unfitted by their very nature to give us information about the differences in mass between systems which have similar spectra (with regard to the "absolute magnitude lines" as well as other characteristics).

When a direct determination of the dispersion in mass among visual binaries has been made—for example, from the trigonometric parallaxes—it will be possible to obtain a valuable control upon the theory which atttributes the spectroscopic phenomena in question to differences in density.

Mount Wilson Observatory
February 16, 1922

ON THE UPPER LIMIT OF DISTANCE TO WHICH THE ARRANGEMENT OF STARS IN SPACE CAN AT PRESENT BE DETERMINED WITH SOME CONFIDENCE[1]

By J. C. KAPTEYN[2] and P. J. VAN RHIJN

ABSTRACT

If both $\phi(M)$ and N_m, the luminosity-curve and the total numbers of stars for each apparent magnitude, were completely known, the star-density could be found for any distance from the sun. The paper deals with the question: What can be reached now that our knowledge of these quantities is limited?

The conclusion is that the incompleteness of our knowledge of $\phi(M)$ is of little importance, but that the contrary is true for N_m. Even if we assume the law of the densities, formula (2), to hold for all values of m, the constants cannot be determined with the accuracy needed for the derivation of the densities at considerable distances. The limits of distance for which the density can be determined within 25 or 40 per cent of its amount were finally found on the supposition that the values of N_m are known to $m=14$ and $m=20$, respectively, and are given in Table XII. With the limit $m=17$, visual, for which we may expect complete values of N_m in the near future, the densities should become pretty reliable for the whole of the domain within which the density exceeds 0.1 of that near the sun.

The conclusions of the present paper must be considered only provisional, mainly for the two following reasons: (*a*) The extinction of light in space has been assumed to be inappreciable; (*b*) No use has been made of a recent substantial improvement in the average parallaxes of the more distant stars.

A more definitive solution now being made at the Laboratory of Groningen will duly take into account these two points. The progress of such a solution being necessarily very slow, the present provisional treatment was undertaken mainly for the purpose of finding the most promising lines for conducting the further treatment of the sidereal problem.

The points which will be successively discussed are as follows:

I. If for any determined part of the sky we know *completely* (*a*) the luminosity-curve $y=\phi(M)$ and (*b*) N_m = the total number of stars of magnitude $m-1/2$ to $m+1/2$ *per ten thousand square degrees*, then we can find the star-density $\Delta(\rho)$ at any distance ρ.

[1] *Contributions from the Mount Wilson Observatory*, No. 229.

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The solution is contained in *Mount Wilson Contribution*, No. 188, formula (13). The difficulty of the problem lies exclusively in the required completeness.

II. Half of the difficulty may be safely ignored, for it can be shown that at present and for a long, long while to come we may certainly adopt the form

$$\phi(M)=\frac{H}{\sqrt{\pi}}e^{-H^2(M-M_0)^2}=e^{p+qM+rM^2} \tag{1}$$

as representing the luminosity-curve completely, that is, for all values of M from $-\infty$ to $+\infty$.

We will here adopt the values of the constants p, q, r, given in *Contribution* No. 188, modified in accordance with the changes in definitions to be explained presently.

III. There thus remains only the difficulty due to our incomplete knowledge of N_m. We will show what limitation is thereby introduced in the solution mentioned in I, and we will further try to show how the limits can be extended by future countings of the fainter magnitudes.

In this inquiry, proceeding from the simple to the more difficult, we will consider three cases:

a) We will *assume* that the formula

$$N_m=\frac{h}{\sqrt{\pi}}e^{-h^2(m-m_0)^2}=2e^{a+bm+cm^2} \tag{2}$$

holds not only for the magnitudes for which we have the necessary data but also for all fainter magnitudes.

b) We will drop the latter supposition, that is, in accordance with the results found thus far, we will adopt the error-curve (2) for the actually observed N_m, but will assume nothing about the fainter magnitudes which have as yet not been investigated.

c) We will consider the most general case, namely, that the actually observed values of N_m themselves are *not* distributed in an error-curve.

IV. Having thus found the limits within which a fairly reliable determination of the arrangement of stars in space can be made, we will introduce what we think must be considered a plausible

supposition, by which these limits, at least for the lower galactic latitudes, can be somewhat farther extended.

Before entering on a discussion of these points it will be necessary to premise the following remarks:

a) Cluster observations by Shapley have fairly well proved the absence of any selective extinction of light in space, at least for distances like those considered in the present discussion. It does not follow that there is no appreciable general extinction of light, and the matter will be carefully considered in our definitive solution. Still, in view of Shapley's observations, we feel less hesitation in making the assumption of no appreciable extinction the basis of this provisional solution.

b) Up to the present, the luminosity-curve has expressed the number of stars of each magnitude contained in a specified volume of space in the neighborhood of the sun. It would certainly have been much more natural and convenient had it been given the form of a true frequency-curve, that is, one showing the fraction of all existing stars belonging to each absolute magnitude. This, of course, is not rigorously possible as long as our observations do not embrace all the stars, even the very faintest in existence. But since the luminosity-curve has been derived (in *Contribution* No. 188) over a range of more than twenty magnitudes and since its coincidence with a normal error-curve throughout this entire interval, which extends far beyond the maximum, is truly astonishing, it would seem very desirable to introduce the *assumption* that the best-fitting error-curve represents the luminosity-curve over its whole extent. In this way it becomes possible to define the luminosity function as a true frequency-curve.

Henceforth we will assume, therefore, that the luminosity-curve, over its whole extent, is perfectly representable by equation (1), which differs from formula (11) in *Contribution* No. 188, only in being divided by the factor $A = 0.0451$, the number of stars per cubic parsec in the neighborhood of the sun. If later on it should appear that this assumption deviates sensibly from the truth, the final results will still be unaffected. The ordinates of the curve will have only to be multiplied by a certain factor, while the densities will be divided by that same factor.

c) The parsec is again taken as a unit of distance, and in accordance with this definition the absolute magnitude of a star will equal its apparent magnitude as seen from a distance of one parsec.

TABLE I*a*

LUMINOSITY-CURVE

(Equation [5])

M	$\phi(M)$	$\log \phi(M)$		
+10	0.00229	7.359		
			470	
+ 9	.00675	7.829		70
			400	
+ 8	.0169	8.229		68
			332	
+ 7	.0365	8.561		69
			263	
+ 6	.0667	8.824		69
			194	
+ 5	.104	9.018		70
			124	
+ 4	.139	9.142		68
			056	
+ 3	.158	9.198		69
			013	
+ 2	.153	9.185		69
			082	
+ 1	.127	9.103		70
			152	
0	.0893	8.951		68
			220	
− 1	.0538	8.731		70
			290	
− 2	.0276	8.441		67
			357	
− 3	.0121	8.084		70
			427	
− 4	.00456	7.657		70
			497	
− 5	.00145	7.160		68
			565	
− 6	.000394	6.595		69
			634	
− 7	.0000914	5.961		69
			703	
− 8	.0000181	5.258		70
			773	
− 9	.00000305	4.485		69
			842	
−10	0.00000044	3.643		

$M = m + 5 \log \pi$

d) Finally, the star-density $\Delta(\rho)$ now means the total number of stars from the brightest to the faintest in one cubic parsec.

Under these stipulations we have in (1), in accordance with *Contribution* No. 188,

$$H=0.2818, \qquad M_0=2.693 \tag{3}$$

$$p=-2.413^{1} \qquad q=+0.4278 \qquad r=-0.07944 \tag{4}$$

$$\log\phi(M)=-1.049+0.1858\,M-0.03450\,M^2 \tag{5}$$

$$M=m+5\log\pi=m-5\log\rho. \tag{6}$$

The numerical values of (5), that is, of the luminosity-curve according to the present definitions, have been tabulated in Table I*a*.

For convenience we give in Table III the values of the constants of *Contribution* No. 188, p. 13, changed to agree with our

TABLE I*b*

log Δ (ρ) (Formulae of Table IV)

log ρ	Galactic Latitude				
	0°	30°	60°	90°	40°–90°
1.0	8.65−10	8.65−10	8.65−10	8.65−10	8.65−10
1.2	.65	.65	.65	.65	.65
1.4	.65	.65	.65	.65	.64
1.6	.65	.65	.65	.65	.61
1.8	.65	.65	.65	.65	.57
2.0	.65	.54	.50	.52	.49
2.2	.65	.46	.42	.45	.40
2.4	.65	.33	.25	.26	.25
2.6	.55	.16	.00	7.95	.03
2.8	.41	7.93	7.67	.51	7.73
3.0	.21	.64	.25	6.95	.32
3.2	7.97	.31	6.75	.27	6.85
3.4	.69	6.92	.15	5.46	.31
3.6	.35	.48	5.48	4.53	5.70
3.8	6.97	5.98	4.72	3.48	.01
4.0	.54	.44	3.88	2.30	4.24
4.2	.07	4.84	2.95	1.00	3.40
4.4	5.54	.19	1.93	....	2.48
4.6	4.97	3.49	0.84	....	1.49
4.8	2.35	2.73	....	....	0.42
5.0	3.68	1.92	....	....	....

[1] Differing from the corresponding value of p in *Contribution* No. 188, p. 15, by the amount $\frac{1}{\text{Mod.}}\log(0.0451)=3.099$.

TABLE II

DISTRIBUTION IN DISTANCE. FRACTIONS OF TOTAL NUMBER OF STARS (W, Equation [12])

$\log \rho$	$m=8$	9	10	11	12	13	14	15	16	17	18	19	20
	Galactic Latitude = 0°												
$-\infty$ to 2.8	0.7779	0.6761	0.5596	0.4382	0.3224	0.2213	0.1409	0.0835	0.0458	0.0231	0.0107	0.0046	0.0018
2.8 to 3.0	.1223	.1590	.1886	.2037	.2002	.1804	.1478	.1107	.0753	.0470	.0268	.0138	.0065
3.0 to 3.2	.0640	.0972	.1344	.1689	.1950	.2048	.1961	.1708	.1366	.0994	.0658	.0400	.0220
3.2 to 3.4	.0256	.0456	.0734	.1085	.1456	.1781	.1994	.2040	.1900	.1610	.1252	.0886	.0570
3.4 to 3.6	.0080	.0164	.0309	.0531	.0833	.1197	.1568	.1864	.2026	.2016	.1825	.1505	.1131
3.6 to 3.8	.0018	.0046	.0100	.0202	.0370	.0617	.0942	.1316	.1670	.1931	.2041	.1975	.1737
3.8 to 4.0	.0003	.0009	.0026	.0058	.0125	.0244	.0439	.0711	.1053	.1428	.1764	.1982	.2038
4.0 to 4.2	.0001	.0002	.0004	.0013	.0032	.0075	.0156	.0296	.0513	.0811	.1166	.1538	.1850
4.2 to 4.4	.0000	.0000	.0001	.0003	.0007	.0017	.0042	.0095	.0192	.0353	.0596	.0916	.1286
4.4 to 4.6			.0000	.0000	.0001	.0004	.0009	.0023	.0055	.0118	.0233	.0417	.0686
4.6 to 4.8					.0000	.0000	.0002	.0004	.0012	.0031	.0070	.0147	.0283
4.8 to 5.0							.0000	.0001	.0002	.0006	.0017	.0040	.0090
5.0 to 5.2								.0000	.0000	.0001	.0003	.0009	.0021
5.2 to 5.4										.0000	.0000	.0001	.0004
5.4 to 5.6												.0000	.0001
	Galactic Latitude 40°–90°												
$-\infty$ to 2.8	0.9190	0.8696	0.8015	0.7161	0.6168	0.5079	0.4001	0.2979	0.2097	0.1397	0.0873	0.0513	0.0281
2.8 to 3.0	.0569	.0860	.1213	.1585	.1920	.2168	.2263	.2207	.1991	.1671	.1298	.0944	.0633
3.0 to 3.2	.0188	.0330	.0545	.0830	.1178	.1551	.1894	.2145	.2267	.2220	.2017	.1706	.1337
3.2 to 3.4	.0044	.0093	.0177	.0317	.0520	.0802	.1143	.1517	.1863	.2131	.2262	.2226	.2042
3.4 to 3.6	.0008	.0018	.0042	.0087	.0168	.0300	.0499	.0774	.1113	.1480	.1837	.2111	.2252
3.6 to 3.8	.0001	.0003	.0007	.0017	.0039	.0081	.0157	.0285	.0480	.0743	.1078	.1446	.1806
3.8 to 4.0	.0000	.0000	.0001	.0003	.0006	.0016	.0036	.0076	.0149	.0271	.0457	.0717	.1045
4.0 to 4.2			.0000	.0000	.0001	.0003	.0006	.0015	.0034	.0071	.0141	.0256	.0437
4.2 to 4.4					.0000	.0000	.0001	.0002	.0005	.0014	.0031	.0067	.0133
4.4 to 4.6							.0000	.0000	.0001	.0002	.0005	.0012	.0029
4.6 to 4.8									.0000	.0000	.0001	.0002	.0004
4.8 to 5.0											.0000	.0000	.0001

modified definitions, that is, with (4) instead of with the values previously given for p, q, r.

The values in Table III lead to the formulae in Table IV. These have been tabulated in Table I*b*.

With these details premised we will now discuss the several points separately.

TABLE III

Parameter	Galactic Latitude				
	0°	30°	60°	90°	40°–90°
h	−8.929	−8.292	−12.320	−17.418	−10.984
k	+5.705	+5.481	+ 9.320	+14.092	+ 8.120
l	−1.366	−1.508	− 2.441	− 3.542	− 2.171

TABLE IV

Gal. Lat.	Density and Distance		
0°	$\log \Delta(\rho) = -3.878 + 2.478 \log \rho$	$-0.593 \; (\log \rho)^2$
30°	$= -3.602 + 2.381$	-0.655
60°	$= -5.351 + 4.048$	-1.060
90°	$= -7.565 + 6.120$	-1.538
40°–90°	$= -4.770 + 3.526$	-0.943

POINT I

As already mentioned, the solution of the problem involved in this case is contained in formula (13) of *Contribution* No. 188.[1] Modified in accordance with what precedes, this formula becomes:

$$N_m = 3.046 \int_0^\infty \rho^2 \Delta(\rho) \phi(m - 5 \log \rho) d\rho. \tag{7}$$

From this equation it is at once clear that given N_m and $\phi(M)$ for *all* values of m and M, we can find $\Delta(\rho)$, and this constitutes in reality the solution of our problem. The derivation involves the solution of the integral equation (7). For the case in which $\phi(M)$ has really the form (1) and N_m the form (2), Schwarzschild has given an elegant solution, which, quoted already in *Contribution* No. 188,

[1] For this formula, in a slightly different form, see *Astronomical Journal*, 24, 27 (No. 566), 1904.

but modified in accordance with our present definitions, is contained in the following formulae:

$$\Delta(\rho)=e^{h+k\log\rho+l(\log\rho)^2} \tag{8}$$

$$\left.\begin{aligned}
l&=\frac{25cr}{r-c}\\
G^2&=-l-25r\\
R&=\frac{G^2(b-q)}{5r}\\
k&=5q-R-\frac{3}{\mu}\\
h&=a-p+\frac{\log G}{\mu}-\frac{R^2}{4G^2}-2.5203\\
\frac{3}{\mu}&=\frac{3}{\text{Mod}}=6.9078.
\end{aligned}\right\} \tag{9}$$

Admitting that (8), which for brevity we will call the Schwarzschild formula, holds for all distances from $\rho=0$ to $\rho=\infty$, it is, of course, easy to determine all such quantities as those which follow (the area is always assumed to be 10,000 square degrees):

N_m = total number of stars of appt. mag. m
$(N_m)_0^\rho$ = the same within the distance ρ
N = total number of stars of all magnitudes together
N_ρ = the same within the distance ρ
$\boldsymbol{M}$ = median = distance within which lie one-half of all the stars
$W=(N_m)_0^\rho/N_m$
$Z=N_\rho/N$.

We find the formulae:

$$N_m=\frac{0.9696\pi\sqrt{\pi}}{\mu G}e^{G^2(\beta^2+\gamma)}=\frac{12.432}{G}e^{G^2(\beta^2+\gamma)} \tag{10}$$

$$(N_m)_0^\rho=WN_m \tag{11}$$

$$W=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{G(\log\rho-\beta)}e^{-z^2}dz \tag{12}$$

$$2\beta G^2=-R-10rm \tag{13}$$

$$\gamma G^2=h+p+qm+rm^2 \tag{14}$$

$$N=\frac{12.433}{\sqrt{-l}}\,e^{h-\frac{\left(k+\frac{3}{\mu}\right)^2}{4l}}\qquad\left(\frac{3}{\mu}=6.9078\right).\tag{15}$$

As a check on the computations of h, k, l, N we have, expressed in terms of a, b, c,

$$N=\frac{\sqrt{\pi}}{\sqrt{-c}}\,e^{a-\frac{b^2}{4c}}\tag{16}$$

$$\log \boldsymbol{M}=-\frac{k+\frac{3}{\mu}}{2l}\qquad\left(\frac{3}{\mu}=6.9078\right)\tag{17}$$

$$N_\rho=ZN\tag{18}$$

$$Z=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{-l}(\log\rho-\log \boldsymbol{M})}e^{-z^2}dz.\tag{19}$$

Introducing the constants of Table III into (12), (13), and (14), we obtain

$$\left.\begin{array}{ll}\text{Gal. lat.}=0^\circ & \text{Gal. lat.}=40^\circ\text{ to }90^\circ\\ \quad G=1.833 & \quad G=2.039\\ \quad \beta=1.560+0.1182m & \quad \beta=1.550+0.0956m\end{array}\right\}\tag{20}$$

with the aid of which equation (12) gives at once an insight into the distribution in distance of the stars of each magnitude m. The results have been summarized for galactic latitudes 0° and for 40° to 90° in Table II. For the medians we find

$$\left.\begin{array}{ll}\text{Gal. lat. }0^\circ & \boldsymbol{M}=39{,}000\text{ parsecs}\\ 40^\circ\text{ to }90^\circ & \qquad 2{,}900\text{ parsecs}\end{array}\right\}\tag{21}$$

Since the Schwarzschild formula does not hold for the smaller distances, results for these have been omitted from the table.

But the supposition that we know $\phi(M)$ and N_m completely, that is, for all values of M and m, is far from being justified. In the present state of science this incompleteness of our knowledge is very serious in the case of N_m, but of little account in the case of $\phi(M)$. The latter part of this statement constitutes the second point.

POINT II

The practical sufficiency of our knowledge of the luminosity-curve is due to the fact, already alluded to, that it has now been found for so large a range of magnitudes, viz., from the very brightest down to about absolute magnitude $M=+9$.

According to (6) a star of apparent magnitude m will be of absolute magnitude $+9$, when its distance is such that

$$\log \rho = \frac{m-9}{5}.$$

If, therefore, we write the equation (7) in the form

$$\frac{N_m}{3.046} = \int_{\log \rho = -\infty}^{\log \rho = \frac{m-9}{5}} \rho^2 \Delta(\rho) \phi(m-5 \log \rho) d\rho + \int_{\log \rho = \frac{m-9}{5}}^{\log \rho = +\infty} \rho^2 \Delta(\rho) \phi(m-5 \log \rho) d\rho \quad (22)$$

the difficulty introduced by incomplete knowledge of the luminosity-curve lies exclusively in the first integral, for the second includes only stars brighter than $M=+9$, for which $\phi(M)$ is known. The difficulty cannot, therefore, be very serious because the first integral involves only stars relatively near the sun. In fact, as will be explained below, the values of $\Delta(\rho)$ within the limits of the first integral may be considered as completely known and its divergence from (8), though considerable, still unimportant for the present question, by a method wholly independent of (7) for all apparent magnitudes to $m=23$ (corresponding to the limit 2.8 for $\log \rho$). It will undoubtedly be a long time before we can expect useful data for N_m beyond $m=23$.

This being so, it is easy to show that there is an overwhelming probability that the first integral in (22) is entirely negligible, at least to the limit $m=23$. We first determine the value of this integral, denoted by V_m, on the supposition that for $M>+9$, $\phi(M)$ is still represented by (1). It is evident that V_m represents the number of stars of apparent magnitude m, absolutely fainter than $M=+9$.

We easily find

$$\frac{V_m}{N_m}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{P} e^{-z^2}dz \tag{23}$$

$$P=\frac{9r-\frac{1}{2}(b-q)-cm}{\sqrt{c-r}}\,. \tag{24}$$

The constants p, q, r are given by (4); a, b, c in *Contribution* No. 188, p. 13.

Table V, computed by the aid of (23), shows that, if the luminosity-curve (1) were correct over its *whole* length, the stars fainter than absolute magnitude +9 would not contribute to the total number N_{23} in the Milky Way more than 1 star in 16,000, or to those in galactic latitudes 40° to 90° more than 1 star in 300. For N_{22}, N_{21}, etc., the contribution is much smaller still.

TABLE V

VALUES OF V_m/N_m

m	Gal. Lat. 0°	Gal. Lat. 40°–90°
18	0.00000050	0.0000113
19	.00000143	.0000414
20	.00000392	.000139
21	.0000104	.00043
22	.0000262	.00121
23	.0000634	.0032
24	.000148	.0076
27	.00143	.0635
30	0.0095	.267

Therefore, even if beyond $M=+9$, the error-curve (1) should begin to fail very seriously to represent the true frequencies of the absolute magnitudes, it will still lead to values of N_m which will be practically correct to magnitude 23 and probably to even much fainter stars.

Suppose for instance that the frequencies, which up to $M=+9$ are represented with such astonishing closeness by the error-curve (1), should, beyond this magnitude, suddenly become tenfold the frequencies furnished by (1); even then the representation would be quite satisfactory for all values of N_m to $m=23$. Of course the

overwhelming probability is that matters are really much more favorable. For we must surely expect that even if the true curve begins to diverge from (1), it will not suddenly show tenfold values. We should rather expect a deviation which for $M=+10$ is still small and only gradually increases, perhaps to very considerable amounts, as we approach fainter and fainter stars. If so, the divergence in the representation of N_m will undoubtedly be still much smaller.[1]

The conclusion from what precedes can only be: as long as observations do not extend beyond $m=23$, formula (7), in which $\phi(M)$ is given by (1), must be considered as valid. Should they ever reach stars fainter than $m=23$, a direct extension of the luminosity-curve beyond $M=+9$ may seem desirable. It is easy to see how this could be accomplished. We should require, however, more extensive data on the very faint stars having very considerable proper motions.

POINT III

It now remains to be seen what result for $\Delta(\rho)$ can be deduced from formula (7) with the aid of our *incomplete* data for the values of N_m. First consider the case:

a) We *assume* the form (2) to hold for all magnitudes from the very brightest down to $m=\infty$, referring for less hypothetical considerations to III*b* and III*c*. The problem evidently is: given, $\phi(M)$ and N_m by the formulae (1) and (2) for all values of M and m; required, $\Delta(\rho)$ for all values of ρ.

The solution has been given above in equations (8) and (9). Although theoretically this solution leaves nothing to be desired, we find in most cases that, for moderately large values of ρ, it becomes practically illusory. The reason for this of course is that existing data do not furnish the values of the constants h, k, l, with sufficient precision.

[1] Perhaps this will not be granted, for of course it is not impossible, *mathematically*, that after $M=+9$, the frequencies instead of continuing to decrease, may begin to increase at such a tremendous rate that the preceding conclusion would no longer hold. Even this possibility must probably be denied. If after $M=+9$ the frequencies should increase at such a rate, the number of very faintly luminous stars near the sun, that is, apparently faint stars with very large proper motion would, I think, exceed what is reconcilable with the observations.

An example, which has a considerable interest of its own, will help to show the difficulty in its true light. As such, we will try to derive the arrangement in space of the stars in the star-cloud between β and γ Cygni. We choose this example because we happen to possess data for part of this region down to photographic magnitude 18.5 (=visual magnitude 17.33). The part in question is that covered by Selected Area (syst. plan) No. 64. The low limit of magnitude is reached by using unpublished results of the Mount Wilson photometric survey together with those in the Harvard Durchmusterung.[1] Simple countings gave the numbers in Tables VI and VII.

TABLE VI

HARVARD DURCHMUSTERUNG

Harv. Photographic Mag.		N'_m for 40′×40′	Comp. by (25)	O−C
Limits	Mean			
8.00 to 8.99	8.5	1	2	− 1
9.00 to 9.99	9.5	7	7	0
10.00 to 10.99	10.5	21	19	+ 2
11.00 to 11.99	11.5	42	49	− 7
12.00 to 12.99	12.5	143	119	+24
13.00 to 13.99	13.5	227	269	−42
14.00 to +14.99	14.5	591	570	+21
Total	....	1032	....	....

TABLE VII

MOUNT WILSON PHOTOMETRIC SURVEY

M.W. Photographic Mag.		N'_m for 15′×15′	N'_m Reduced to 40′×40′
Limits	Mean		
12.5 to 13.5	13.0	20½	146
13.5 to 14.5	14.0	43½	309
14.5 to 15.5	15.0	97½	693
15.5 to 16.5	16.0	200½	1426
16.5 to 17.5	17.0	363½	2584
17.5 to 18.5	18.0	677½	4817
Total	....	1403	....

[1] *Harvard Annals* 101.

These tables do not contain even approximately all the stars in the two catalogues. The faintest stars measured on the Harvard plates are 15.75 (phot.), those on the Mount Wilson plates 18.86 (phot.). Stars fainter than 14.99 and 18.50, respectively, were excluded because we cannot be sure that the catalogues are absolutely complete below these limits.

We have first of all to reduce the magnitudes to the same scale and to the same area. The latter reduction has already been made in the last column of Table VII. To find the scale difference the observed numbers N'_m in Table VI were represented by an equation of the form of (2). The result may be written

$$\log N'_m = 2.074 + 0.370(m - 12.5) - 0.0145(m - 12.5)^2. \qquad (25)$$

The values computed by means of this formula are in the fourth column of Table VI; the fifth shows the residuals O−C, whose changes of sign are all that can be desired. With the aid of (25) the Harvard magnitudes were calculated for which the numbers N'_m become equal to the first three values of N'_m in Table VII, viz., 146, 309, 693. The results are 12.75, 13.68, and 14.78, whence we conclude:

	Mt. W.−Harv.
Harv. mag. 12.75 = Mt. W. mag. 13.0	$+0^{\text{m}}.25$
Harv. mag. 13.68 = Mt. W. mag. 14.0	+0.32
Harv. mag. 14.78 = Mt. W. mag. 15.0	+0.22

There is no indication in the differences Mt.W.−Harv. of change with the magnitude, and we adopt

$$\text{mag. Mt. W.} - \text{mag. Harv.} = \text{Const.} = +0^{\text{m}}.26. \qquad (26)$$

By means of (26) the values in Table VII can now be reduced to the scale of Table VI.

A further reduction from the photographic to the visual scale is necessary because the luminosity-curve defined by (1) and (4) gives the frequencies for the visual magnitudes. For this we have used the corrections given in *Groningen Publications*, No. 27, p. 42. The first five columns of Table VIII show the complete reduction.

TABLE VIII

N'_m FOR 40′×40′

Photographic Scale		Vis. Scale Harv.	Ampl.	$\overset{m}{\text{H.S.}}$	N'_m Ampl. 1. Mag.	Computed		O−C	
Harv.	Mt. W.					(27)	(28)	(27)	(28)
8.00 to 8.99	....	7.64 to 8.61	0.97	8.125	1	2	3	− 1	− 2
9.00 to 9.99	....	8.62 to 9.57	.95	9.095	7	7	8	0	− 1
10.00 to 10.99	....	9.58 to 10.52	.94	10.05	22	22	22	0	0
11.00 to 11.99	....	10.53 to 11.47	.94	11.00	44	55	54	− 11	−10
12.00 to 12.99	....	11.48 to 12.41	.93	11.945	154	134	127	+ 20	+27
13.00 to 13.99	....	12.42 to 13.34	.92	12.88	247	296	281	− 49	−34
14.00 to 14.99	....	13.35 to 14.28	.93	13.815	635	629	598	+ 6	+37
....	12.5 to 13.5	11.71 to 12.67	.93	12.175	157	163	155	− 6	+ 2
....	13.5 to 14.5	12.64 to 13.58	.94	13.11	329	358	340	− 29	−11
....	14.5 to 15.5	13.58 to 14.51	.93	14.045	745	748	716	− 3	+29
....	15.5 to 16.5	14.51 to 15.45	.94	14.98	1517	1475	1439	+ 42	+78
....	16.5 to 17.5	15.45 to 16.39	.94	15.92	2749	(2751)	2793	(− 2)	−44
....	17.5 to 18.5	16.39 to 17.33	0.94	16.86	5124	(4835)	5176	(+289)	−52

The fifth and sixth columns give the observational data fully reduced for the present discussion. The numbers in the cloud were first compared with the corresponding numbers for the average Milky Way. It turned out at once that these numbers are all but perfectly proportional, the cloud having about 2.39 times the number for the average Milky Way. This is shown by the seventh column of the table which gives the carefully interpolated values for the Milky Way according to *Groningen Publication*, No. 27, reduced to an area of $40' \times 40'$ and multiplied by 2.39. The agreement with the observed cloud numbers in the sixth column is as good as can be expected. The numbers in the Milky Way in *Groningen Publication*, No. 27, do not extend beyond magnitude 16. For these very faint magnitudes the table gives in parentheses the values furnished by

$$\log N'_m = -4.516 + 0.7242m - 0.0141m^2 \tag{27}$$

which best fits the numbers of *Groningen Publication*, No. 27, reduced to $40' \times 40'$ and multiplied by 2.39.

On the assumption made in this paragraph, that the formula holds for *all* magnitudes, we conclude that at all distances the star-densities in the cloud are 2.39 times the corresponding densities in the average Milky Way.

Meanwhile we can represent the observed numbers quite as well by a formula somewhat different from (27); for instance, by

$$\log N'_m = -3.951 + 0.6334m - 0.0106m^2. \tag{28}$$

The representation by this formula is also shown in Table VIII.

By means of the constants in (27) and (28) and those of the luminosity-curve (5), we can at once pass from the distribution of the values N_m to that of the densities by means of formulae (8) and (9). We thus obtain

	Solution (27)	Solution (28)	
h	-8.060	-3.488	
k	$+5.719$	$+2.678$	(29)
l	-1.375	-0.882	

The two solutions represent rather widely different arrangements in space. The best way of showing the difference in

compact form will perhaps be to compute for both, by formulae (16) and (17),

N = total number of stars for a field of $40' \times 40'$, and

$\boldsymbol{M}$ = median distance below which lie just half of all the stars.

Introducing the values (29), we find,

$$\begin{array}{lll} & \text{Solution (27)} & \text{Solution (28)} \\ \boldsymbol{M} \ldots\ldots\ldots\ldots & 39{,}000 \text{ parsecs} & 272{,}000 \text{ parsecs} \\ N_0^\infty \ldots\ldots\ldots & 579{,}000 \text{ stars} & 3{,}700{,}000 \text{ stars} \end{array} \Bigg\} \quad (30)$$

Identical data thus lead to two solutions for the arrangement in space of the stars in the cloud. In the second, judging by the median values, the star-distances are 7 times those of the first solution. The total numbers[1] are as 1 to 6.5.

Evidently, therefore, the data at present available do not lead to a reliable determination of the real structure of the cloud. The reason clearly lies in the fact that the data of Table VIII are insufficient for a good determination of the constants for N_m in equation (2), and it is readily understood that it is mainly the uncertainty in the position of the maximum to which the failure of the solution is due.

In the notation of (2), this maximum lies at the magnitude

$$m_0 = -\frac{b}{2c} \tag{31}$$

whence for

$$\left.\begin{array}{ll} \text{Solution (27)} & m_0 = 25.68 \\ \text{Solution (28)} & m_0 = 29.88 \end{array}\right\} \quad (32)$$

a difference of over 4 magnitudes!

We are thus led to consider the question as to how far the observed data must be extended in order to obtain a determination of $\Delta(\rho)$ which will satisfy moderate demands at least.

To answer this it will be necessary to determine: (*a*) how the precision of m_0 (prob. err. ρ_{m_0}) depends on the precision of the

[1] Since the total number of stars resulting from the two solutions differs enormously, it might seem worth trying to decide between them by measuring the total light of the cloud. To test the feasibility of such a plan I compared the values of the total light and found that the amounts involved in (27) and (28) are as 1.000 to 1.025. Practically, therefore, the idea is worthless, at least with the photometric means at present available.

counts N_m; (b) how the precision of the median $\boldsymbol{M}$ depends on the precision of m_0.

As for (a): Consider first the somewhat easier question: What will be the accuracy of m_0 obtained from observed values of N_m for three, favorably chosen, equidistant magnitudes

$$m_1-\sigma; \qquad m_1; \qquad m_1+\sigma? \tag{33}$$

For brevity call this the simplified problem. Since a rough estimate is all that is wanted, assume that the probable error of log N_m for these three magnitudes is constant and equal to r. When this simplified problem has been solved it will be easy to estimate roughly the accuracy obtainable by the use of all values of N_m.

In the Appendix is derived the formula which shows the relation between the probable errors ρ_{m_0} and r, namely,

$$\rho_{m_0}=\pm\frac{r\sqrt{2}}{4\sigma^2c\mu}\sqrt{12(m_0-m_1)^2+\sigma^2}\,. \tag{34}$$

The value of r is readily determined by means of the deviations of the observed log N_m from a close-fitting curve. In the case of the star-cloud the number of observed stars brighter than 12 is rather small. This being the case, consider only the Mount Wilson observations in Table VIII, which show a better agreement *inter se* than the Harvard values. For the best-fitting curve I used a third solution, intermediate between (27) and (28), for which the constants are:

$$\begin{array}{llll} a=-10.019 & h=-6.038 & & \\ b=+1.570 & k=+4.289 & (\text{area}=40'\times 40') & (35) \\ c=-0.0294 & l=-1.165 & & \\ m_0=26.73 & \text{Median } \boldsymbol{M}=63{,}800 \text{ parsecs} & & \end{array}$$

The value found for the probable error is $r=\pm 0.008$. Experience gained in other cases indicates that this value is accidentally low and that a better one would be

$$r=\pm 0.012. \tag{36}$$

For the fundamental magnitudes we choose the values (see Table VIII)

$$12.18, \qquad 14.52, \qquad 16.86, \tag{37}$$

then for the computation of ρ_{m_0} by (34)

$$\left.\begin{array}{ll} m_1 = 14.52 & \sigma = 2.34 \\ m_0 = 26.73 & c = -0.0294 \end{array}\right\} \tag{38}$$

which lead to

$$\rho_{m_0} = \pm 2.6 \text{ mag.} \tag{39}$$

Starting from solution (27) instead of (35), we find nearly the same result.

As for (*b*): We learn how the accuracy of the median $\boldsymbol{M}$ depends on that of m_0 by comparing the values of m_0 and $\boldsymbol{M}$ in the three solutions (27), (28), (35). Adding a fourth solution, we have altogether

$$\left.\begin{array}{cccc} m_0 & & \log \boldsymbol{M} & \\ \hline 25.68 & & 4.592 & \\ & 1.05 & & 0.213 \\ 26.73 & & 4.805 & \\ & 1.05 & & 0.203 \\ 27.78 & & 5.008 & \\ & 2.10 & & 0.426 \\ 29.88 & & 5.434 & \end{array}\right\} \tag{40}$$

Evidently log $\boldsymbol{M}$ changes very nearly proportionally to m_0. In fact

$$\log \boldsymbol{M} = -0.544 + 0.2\, m_0. \tag{41}$$

Therefore

$$\rho_{\log \boldsymbol{M}} = 0.2\, \rho_{m_0}; \tag{42}$$

consequently, adopting the values (39),

$$\rho_{\log \boldsymbol{M}} = \pm 0.520 \text{ (simplified problem)}. \tag{43}$$

It is now easy to see what can be gained by future extension of the counts to fainter magnitudes.

Table IX shows how the probable errors of both m_0 and log $\boldsymbol{M}$ change with increase in the limiting magnitude. In accordance with what precedes, we adopted ± 2.6 as the probable error of m_0 in the case of the limiting magnitude 17.5, i.e., for limiting $N_m = 17.0$. The remaining probable errors were then computed by formulae (34) and (42). The results are summarized in Table IX.

Hence, if we were able to photograph the stars to magnitude 20.5 (visual scale), the probable error of log M in the simplified problem would be diminished to 0.176, which, roughly speaking, is equivalent to a probable error of about 42 per cent in the numerical value of M. I believe the error would be further lowered to less than 30 per cent, if the calculation of m_0 were based on all the available data for N_m instead of merely on those for the three fundamental magnitudes used for the simplified problem.

From an absolute point of view this precision may not seem to be very satisfactory. Still if it can be reached—and at present this is hardly subject to doubt—an enormous step in advance will have been made, for it means that the median parallax will have been determined with a probable error of about $\pm 0\rlap{.}''00001$.

TABLE IX

PROBABLE ERROR OF m_0 AND LOG M (SIMPLIFIED PROBLEM)

Limiting Magnitude	Fund. Mags.			m_1	σ	m_0 Sol. (35)	ρ_{m_0}	$\rho_{\log M}$
17.5	12.0	14.5	17.0	14.5	2.5	26.73	2.60	0.520
18.5	12.0	15.0	18.0	15.0	3.0		1.73	.346
19.5	12.0	15.5	19.0	15.5	3.5		1.21	.242
20.5	12.0	16.0	20.0	16.0	4.0		0.88	.176
21.5	12.0	16.5	21.0	16.5	4.5		0.66	.132
22.5	12.0	17.0	22.0	17.0	5.0		0.51	0.102

Moreover, the accuracy can be greatly increased by taking several fields instead of but one.

The preceding considerations on the structure of one small region may of course be applied without change to the investigation of larger parts of the sky, for instance, to the Selected Areas for different zones of galactic latitude. The only difference is that we will obtain a much greater accuracy.

From the data of *Groningen Publication*, No. 27, Table V, we have found the values of m_0 for different galactic latitudes (β) entered in the second column of Table X. The other quantities in the table were obtained in the manner already explained.

In this table the probable errors are given, in the simplified problem, as they would be found for a single region containing the same number of stars as that in the case treated above. In con-

formity with (39), ρ_{m_0} for $\beta=0$ and lim. mag. 17.5 was put at ± 2.6. For the rest the computation was made with the aid of (34) and (42).

As was to be expected, the values of m_0 run somewhat irregularly. We therefore adopted the smoothed values of the third column, which, with the exception of that for $\beta=20°$, represent the observations practically as well as those of the second column.

By using for each region all the material instead of the three values of N_m of the simplified problem, the probable error will be diminished. If further for each galactic latitude we take the greater part or all of the Selected Areas, the probable error will again be much reduced. Finally, a further important reduction

TABLE X

Probable Error in Simplified Problem for a Region 15′×15′

β	m_0	Adopted m_0	c	Lim. Mag. 17.5		Lim. Mag. 20.5	
				ρ_{m_0}	$\rho_{\log M}$	ρ_{m_0}	$\rho_{\log M}$
0°........	26.7	26.7	−0.0294	±2.6	±0.52	±0.88	±0.176
20........	27.2	25.6	− .0297	2.4	.48	.79	.158
40........	23.1	22.9	− .0333	1.6	.32	.51	.102
60........	19.1	19.9	− .0407	0.85	.17	.24	.048
80........	18.2	17.9	−0.0486	±0.45	±0.09	±0.11	±0.022

can be made by taking into account the very extensive data for the brighter stars. In order to see more clearly what can be gained, some estimate must be made as to the effect of all these reducing causes. The matter certainly deserves careful study. We have not yet made such a study, but estimate provisionally that it should be easy to reduce the probable errors of Table X to one-third.[1]

For clearness we may express roughly the accuracy corresponding to this assumption in percentages of the whole. The results for the median M are thus found to be as given in Table XI.

Granting the suppositions made in III*a*, and that we are in possession of data for N_m to magnitude 20.5, we conclude that we can obtain what may be considered a good insight into the

[1] The comparison of the values m_0 in the second column of Table X with the smoothed values in the third leads at once to a precision of about this amount, notwithstanding the fact that the data of *Groningen Publication*, No. 27, extend but little beyond magnitude 14.5 visual.

distribution of the stars in all galactic latitudes. With data to 17.5 alone (Mount Wilson Survey) this can be said only for the regions of high galactic latitude. Below latitude 40° the solution becomes rapidly less reliable.

Remark.—Meanwhile we need an extension of star-counts for regions near the galactic pole quite as much and probably more than for the lower latitudes, for just these observations of N_m in the higher latitudes will lead to a judgment on the general validity or non-validity of the analytical form (2). This follows because of the lower value of m_0 (Table X). Suppose for instance

TABLE XI

PROBABLE ERROR OF MEDIAN M IN PERCENTAGE OF ITS AMOUNT

Gal. Lat.	Limiting Magnitude	
	17.5	20.5
0°	41	13
20	38	12
40	25	8
60	13	4
80	7	1.5

that we obtain counts to apparent magnitude 20.5. In high latitudes the curve for N_m will then become known considerably beyond the maximum, that is, for a very considerable fraction of the whole, and a fraction very much greater than that covered in the lower latitudes. In case formula (2) is confirmed for this large part of the curve in the region of the galactic pole, we will use the same analytical form with much greater confidence for the lower latitudes also.

III*b*. We know by observation that as far as magnitude 14 the values of N_m are well represented by (2); for the stars fainter than 14 we know nothing of the sort.[1] What results will it be possible to obtain from this knowledge?

Schwarzschild's solution, which is based on the supposition that *all* values of N_m fall on the curve (2), led to formula (8) which, but for the observation errors, would yield a knowledge of $\Delta(\rho)$ for *all*

[1] We take the limit $m=14$ because it represents what at present is really well known. The reasoning, of course, holds for any other limit.

distances. This solution is not now generally applicable and we must look for another. Such a solution, based on quite different principles, which I will call solution K, has already been given in *Groningen Publication*, No. 11, and has also been applied in *Contribution* No. 188. It has there been carried through for galactic latitude 40° to 90°, and shows the very gratifying fact, which we had no right to expect beforehand, that, but for small distances (where Schwarzschild's solution is confessedly wrong) it agrees wonderfully well with the Schwarzschild solution. With the exception referred to, therefore, solution (7) represents all the data, those of *Contribution* No. 88 as well as the values of N_m in a way that leaves hardly anything to be desired.

Therefore, notwithstanding what has just been said, the Schwarzschild solution (8) must still be considered the most plausible to be obtained with the aid of the data at present available. This must be owing to the fact that further values of N_m, particularly those for the most influential magnitudes 15, 16 . . . . , do not deviate very much from the curve (2). Still this does not prove that formula (8) gives even an approximate solution for $\Delta(\rho)$ at very great distances. We see, for instance, from Table II, that according to (8) there are, in high galactic latitudes, hardly any stars beyond $\log \rho = 4.2$. As a consequence thereof, suppose that between $\log \rho = 4.0$ and $\log \rho = 4.2$, the density is a hundred fold the value given by (8); the total numbers of stars N_{14} will be changed by about 1 per cent only. For the brighter magnitudes the change would be still smaller. Evidently, therefore, nothing can be concluded from our data about the densities at such great distances. They may or may not agree with formula (8); we have no means of judging. The matter would be different if we had good counts (N_m) for very much fainter stars. The question thus presents itself: To what distance can the densities computed by (8) be accepted with some confidence? Or to be more precise: To what distance can we trust the values of $\Delta(\rho)$, found by (8), to be correct within 25 and 40 per cent, respectively, of their values?

In considering this question we will *assume* that what was found in *Contribution* No. 188 for galactic latitudes 40° to 90° will also be

found for other galactic latitudes, viz., complete agreement of solution K with solution (8). It seems to me very probable that such will turn out to be the case. As solution K is independent of the values of N_m for $m>14$ we may thus accept formula (8) for $\log \rho \leqq 2.8$[1] (excepting of course the small distances).

The question under consideration comes to this: How much can we change the densities (8) for $\log \rho > 2.8$, without spoiling the representation of the values of N_m to $m=14$? Suppose that, for galactic latitudes 40° to 90°, we multiply the densities for

$$\log \rho = 2.8 \text{ to } 3.0 \text{ by } F_1,$$
$$\log \rho = 3.0 \text{ to } 3.2 \text{ by } F_2, \text{ etc.}$$

Then the numbers in Table II (second part) for $\log \rho = 2.8$ to 3.0 will all be multiplied by F_1; those of the following line by F_2, etc.

The total number N_8 (see Table II) will thus be changed to

$$N_8(0.9190+0.0569F_1+0.0188F_2+0.0044F_3+0.0008F_4+0.0001F_5) \quad (44)$$

and, in order that the N_8 may still be represented, the sum must still equal N_8; or better, the expression in parentheses must equal 1.00. We also find similar expressions for magnitudes 9, 10, ... 14.

The question thus becomes: What is the smallest distance for which we can take for any one of the factors F, a value as high as 1.25 or as low as 0.75 (or, respectively, 1.40 and 0.60) without making it impossible to determine the remaining factors in such a way that the seven equations of condition are still bearably well satisfied. For "bearably well satisfied" we have used the condition that the range in the residuals should not exceed 10 per cent.

Suppose, therefore, having assumed the constant F for distance R to be 1.25, that we solve our equations of condition and obtain indeed a "bearable" set of residuals. Will this prove that for distance R a density deviating 25 per cent from (8) is admissible? By no means. For as a rule the values found for the other constants will be impossible or inadmissible because they lead to negative values of $\Delta(\rho)$ or to values running too wildly for the regularly increasing distances.

[1] In *Contribution* No. 188 this is the limit to which solution K extends.

In order to obtain an acceptable solution we must evidently introduce the condition that the values of F shall run smoothly for regularly increasing distances. This condition is pretty vague and introduces a certain degree of arbitrariness, but I think that there is really no serious objection to it. Moderate differences in the form of the curve will probably change the results little as long as we leave two or three constants indeterminate, to be found in a way best satisfying the conditions of the problem.

Before stating the form adopted for the factors F, consider the general course they must follow. From the value 1.00 for $\log \rho = 2.8$ they must either increase or decrease. Consider only the first case, because by satisfying this we more than satisfy the other. The initial increase cannot indefinitely continue because if all the factors are greater than 1.00, the total numbers N_m will all be increased. The values of N_m cannot well be represented, therefore, unless values of $F > 1.00$ are compensated by later factors < 1.00. The factors must therefore pass through a maximum, after which decrease sets in and they soon become 1.00. We assume that there is but one maximum. Consider now the various curves representing various hypotheses for the factors F. Take more curves for which the value of the maximum is respectively $F_{max} = 1.25$, $F_{max} = 1.30$, $F_{max} = 1.35$, ... and let R, R_1, R_2, ... be the values of the distance for which $F = 1.25$.

Then we think that generally

$$R < R_1 < R_2 \ . \ . \ . \ .$$

That is, the factor $F = 1.25$ will be reached sooner by that curve for which $F = 1.25$ is a maximum.[1]

Since we wish to find the smallest distance at which F can become equal to 1.25, we consider only the latter case, that is, find the minimum distance, R, at which F can reach 1.25 as a maximal value, without being led to an "unbearable" representation of N_m.

This being premised, we now give the mathematical form adopted for F:

[1] We have not tried to find the demonstration. We have been content to verify its truth further on by means of the data in Table X.

$$\left.\begin{array}{l}\text{A. For the case, maximum factor} = 1.25\\ \quad\text{Before the maximum, } F = 0.96 + 0.29\, e^{-h^2x^2}\\ \quad\text{After the maximum, } F = 1.25\, e^{-p^2x^2}\end{array}\right\} \tag{45}$$

$$\left.\begin{array}{l}\text{B. For the case, maximum factor} = 1.40\\ \quad\text{Before the maximum, } F = 0.96 + 0.44\, e^{-h^2x^2}\\ \quad\text{After the maximum, } F = 1.40\, e^{-p^2x^2}\end{array}\right\} \tag{46}$$

in both of which

$$x = \log \rho - \log R. \tag{47}$$

There remain, therefore, the three constants R, h, p, to be determined in such a way that R corresponds to the smallest distance for which the values of N_m can be represented with residuals not exceeding 10 per cent. This form for the factors F gives perfect smoothness throughout. It is that which a priori seems most plausible. Beyond the maximum, $\Delta(\rho)$ is evidently again represented by a form like (8), which represented the densities for the smaller distances with such perfection.

It does not, however, start at $\log \rho = 2.8$ with the initial value 1.00, as premised. The reason is that the small discontinuity thus introduced at $\log \rho = 2.8$ is of no real importance, while the representation of N_m is appreciably better.

Substituting the expressions for F into (44) and equating the result to the successive values of N_m we obtained a number of equations of condition.

In solving them we generally began by assuming two values for R, for each of which these equations were solved for h and p in such a way that practically the best possible representation of N_m(to $m = 14$) was obtained. The range of the residuals was then found for both cases. The value of R for which the range is 10 per cent could then usually be found by interpolation. If not, the computation was repeated for a third value of R. In this way were found all the values entered in Table XII.

In all cases, with the exception of the third for $m = 14$, the range of the residuals in N_m is nearly 10 per cent. In the exceptional case it is only 6.5 per cent. In fact, in this case, the knowl-

edge of N_m to $m=14$ takes us little beyond what was reached by solution K.

From Table XII we may be pretty sure, now that we know N_m with some precision to $m=14$, that in the Milky Way there will be in the densities as computed by (8), no error of 25 per cent up to log $\rho=3.1$ and no error of 40 per cent up to log $\rho=3.2$. As soon as we have reliable knowledge of N_m to $m=20$ these limits will rise to log $\rho=3.8$ and 4.0, respectively. We further learn from the table that by extending our knowledge of N_m from $m=14$

TABLE XII

LIMITS OF RELIABILITY OF $\Delta(\rho)$ BY (8)

GAL. LAT.	N_m KNOWN TO $m=14$				GAL. LAT.	N_m KNOWN TO $m=20$			
	F_{max}	log R	h	p		F_{max}	log R	h	p
0°	1.25	3.1	7.0	1.66	0°	1.25	3.8	4.0	1.66
0	1.40	3.2	6.0	2.70	0	1.40	4.0	3.7	3.6
40 to 90	1.25	2.9	......	2.20	40 to 90	1.25	3.4	4.0	2.05
40 to 90	1.40	2.9	......	3.0	40 to 90	1.40	3.5	4.0	3.73

to $m=20$ we will extend the limits to which we can reliably determine the structure of the system:

$$\left.\begin{array}{ll}\text{In the Milky Way} & 5.6 \text{ fold } (\log=0.75) \\ \text{In Gal. Lat. } 40° \text{ to } 90° & 3.5 \text{ fold } (\log=0.55)\end{array}\right\} \quad (48)$$

In the case of the Mount Wilson Survey, which will yield N_m to $m=17$, the gain will be about half this amount.

III*c*. If finally, having extended the counts N_m to some higher magnitude, we find that the values of N_m no longer fit an error-curve like (2), then evidently formula (8) will no longer hold for the densities. In this case we must fall back on formula (7). We will thus find different values for the densities $\Delta(\rho)$. We may certainly expect, however, that, as long as these values do not deviate very widely indeed from the densities furnished by such a formula as (8), our considerations on the limits of reliability will still practically hold. For the present, therefore, we may safely rest content with the results just obtained.

POINT IV. EXTENSION OF THE LIMITS OF TABLE XII

There is, we think, reason to assume that the limits shown in Table XII give rather too unfavorable an impression of the real reliability. The reason is that the deviation, having once reached 25 (or 40) per cent, will grow no further, or at least not much further before it begins to diminish. This it will continue to do until it becomes *zero* and begins to increase again and, possibly, go on increasing indefinitely (on the other side). That is to say, beyond the limits assigned in Table XII the deviation from (8) for a certain time will still be mostly very moderate or small. Still much more important than this is a fact which will appear in a further publication, by one of us, in which a formula is derived theoretically which in the last analysis is an expression of the differential quotient $d\Delta(\rho)/d\rho$ as a function of ρ and $\Delta(\rho)$. The integration of this differential equation, if the equation could be accepted as definitely proved, would give $\Delta(\rho)$ as a function of ρ for all distances. As it is, the equation is verified by observation as far as observation can fairly be relied on.

We thus have not only reliable values of $\Delta(\rho)$ up to the limits of Table XII, but also to these same limits, reliable values of $d\Delta(\rho)/d\rho$. This proves that we have a right to extend the use of the formula which yielded $\Delta(\rho)$ considerably beyond the limits of Table XII.

There is a third consideration which will enable us to extend the limits of Table XII with some probability of reliable results. In the publication just alluded to it will be shown that the equidensity surfaces are at least approximately similar ellipsoids. If we suppose that this form holds somewhat beyond the limits for which we have fair certainty, then the limits of Table XII will again be extended, at least for the lower galactic latitudes.

The three considerations (of which the second is the most important) together make us believe, that as soon as we have good data for N_m down to magnitude 17, which will be very shortly, we shall be able to find fairly reliable values of the $\Delta(\rho)$ for the entire region near the center for which $\Delta(\rho)$ exceeds one hundredth of density near sun, that is, for the whole of the space represented in Fig. 2, *Contribution* No. 188.

CONCLUDING REMARK

The conclusions arrived at in this paper may perhaps best be seen in their true bearing by comparing the limits within which we can find fairly reliable results for the structure of the stellar system by using different sets of data. Of course Table XIII claims no other merit than that of furnishing the means of a rough insight into what can be obtained at present or in the very near future.

TABLE XIII

LIMITS IN PARSECS WITHIN WHICH THE STRUCTURE OF THE STELLAR SYSTEM CAN BE FOUND

Method	Galactic Latitude		
	0°–20°	40°–90°	0°–90°
1. Direct parallax determination.	. . . .	. . . .	50
2. Parallactic motion, now well known, to $m=10$	320	240	300
3. The same, to $m=13$.	(830)	(610)	(720)
4. Parallactic motion, stars to $m=10$ and $\mu=0''.01$	400	320	360
5. The same together with N_m (the latter to $m=14$).	1600	800	1200
6. The same, N_m to $m=17$	(4000)	(1600)	(2800)
7. Extension according to Point IV, N_m to $m=14$.	3000	1000	2000
8. The same, N_m to $m=17$.	(8000)	1700	$\Delta=\Delta_0/100$

NOTES TO TABLE XIII

The limits in parentheses have not yet been reached but may be attained in two or three years.

1. The magnificent results lately obtained in direct parallax determinations by means of long-focus instruments show probable errors of the order of $0''.01$. Even if there were demonstrably no systematic errors left, a probable error of $0''.01$ would imply that we cannot derive from them the structure of the system beyond the limit of the table. Since the absence of systematic error cannot be claimed, the limit of the table is probably rather high.

2, 4. The table gives simply the mean distances corresponding to the mean parallaxes in *Groningen Publication*, No. 29, and in a paper not yet published.

3. Simple repetition of the parallax plates obtained at the Mount Wilson Observatory some five or more years after their first exposure, would give excellent values for the parallactic motion of the stars of magnitude 13. Similar repetition of the parallax work done in America on Boss stars by other observatories with long-focus instruments would also contribute to an important degree.

5, 6. From Table XII of the present paper.

7, 8. According to Point IV of present paper. In the last column of No. 8, $\Delta=\Delta_0/100$ means that for all latitudes the densities are fairly reliable for the whole of the space where the density is greater than 0.01 of that near the sun (compare Fig. 2, *Contribution* No. 188).

All the results will be greatly strengthened as soon as we know N_m to $m=20$. At the same time this will enable us to make good estimates for still greater distances.

APPENDIX

POINT III*a*. ON THE SIMPLIFIED PROBLEM

Given, r = probable error of each of the quantities $\log N_1$, $\log N_2$, $\log N_3$, required, ρ_{m_0} = probable error of the constant m_0.

By formulae (31) and (2) of the text

$$\log N_m = \mu a - 2\mu m_0 cm + \mu cm^2 \qquad (\mu = \text{mod.}). \tag{a}$$

Hence

$$\left.\begin{aligned} \log N_1 &= \mu a - 2\mu m_0 c(m_1 - \sigma) + \mu c(m_1 - \sigma)^2 \\ \log N_2 &= \mu a - 2\mu m_0 c m_1 + \mu c m_1^2 \\ \log N_3 &= \mu a - 2\mu m_0 c(m_1 + \sigma) + \mu c(m_1 + \sigma)^2 \end{aligned}\right\} \tag{b}$$

from which for m_0

$$m_0 - m_1 = \frac{(H+1)\sigma}{2(H-1)} \tag{c}$$

in which

$$H = \frac{\log N_2 - \log N_1}{\log N_3 - \log N_2}. \tag{d}$$

Consequently

$$\rho_{m_0} = \frac{\sigma}{(H-1)^2}\rho_H \tag{e}$$

in which the probable error ρ_H must be found from (*d*). Since we assume

$$\text{Probable errors of } N_1,\ N_2, \text{ and } N_3 \text{ each} = r, \tag{f}$$

we obtain as in the text

$$\rho_{m_0} = \pm \frac{r\sqrt{2}}{4\sigma^2 c\mu}\sqrt{12(m_0 - m_1)^2 + \sigma^2}\,. \tag{g}$$

LEIDEN, HOLLAND
August 1921

COMPARISON STARS FOR NOVA PERSEI, No. 2
A CORRECTION

Father Hagen has called my attention to an error in the identification of one of the comparison stars for Nova Persei, No. 2, whose magnitudes are given in *Mt. Wilson Contr.*, No. 192; *Astrophysical Journal*, **52**, 183, 1920. The star in question is No. 57 of his sequence. An uncertainty in the position and brightness was noted during the measurement of the photographs, but was overlooked when the MS was prepared several months later.

The confusion arose from the fact that the declination coordinates of stars 57 and 62 ($+12'.6$ and $+11'.5$, respectively) are interchanged in the Chart and Catalogue of Father Hagen. The star actually observed here is close to the catalogue position of No. 57, but is much fainter than that object. Neither No. 57 nor 62 has been measured on the Mount Wilson photographs; No. 57, in fact, is just outside the limit of distance from the center of the plate usually adopted. The visual magnitudes on the Mount Wilson scale are readily found, however, by reducing Father Hagen's values to that system. The results are

No. 57 $12^{M}.7$; No. 62 $12^{M}.8$

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